TOPICAL REVIEW

Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations

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Abstract
The Painlevé property is closely connected to differential equations that are integrable via related iso-monodromy problems. Many apparently integrable discrete analogues of the Painlevé equations have appeared in the literature. The existence of sufficiently many finite-order meromorphic solutions appears to be a good analogue of the Painlevé property for discrete equations, in which the independent variable is taken to be complex. A general introduction to Nevanlinna theory is presented together with an overview of recent applications to meromorphic solutions of difference equations and the difference and q-difference operators. New results are presented concerning equations of the form $w(z+1)w(z-1) = R(z, w)$, where R is rational in w with meromorphic coefficients.

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Contents

1. Introduction

Ordinary differential equations whose solutions have especially simple singularity structure are known to have remarkable integrability properties. The Painlevé equations are particularly important in this context. These six equations (denoted P_I–P_{VI}) possess the Painlevé property, i.e., all movable singularities of all solutions are poles, and have been known for over a century. Each of the Painlevé equations can be written as the compatibility condition for a related linear problem, a so-called *iso-monodromy* problem, which underlies the integrability of the equation.

The discrete equation

$$x_{n+1} + x_n + x_{n-1} = \frac{an+b}{x_n} + c, \tag{1}$$

where a, b and c are constants, appeared in studies of a partition function in quantum gravity [12, 17, 32, 41, 101]. It was realized that equation (1) is very special, in particular, it is the compatibility condition for a linear problem, which was studied in [70], and it has a simple continuum limit to the first Painlevé equation, P_I [17, 25, 41]. It would appear natural to call equation (1) a discrete Painlevé equation. Equation (1) had previously appeared in works on the coefficients in the three-point recurrence relations of certain classes of orthogonal polynomials [33, 71, 122]. It was Magnus [84] who explicitly noted that a discrete Painlevé equation arises in this context.

Soon other discrete Painlevé equations were discovered as reductions of integrable lattice equations [94] and by looking for discrete isomonodromy problems [31, 74]. But what exactly is a discrete Painlevé equation? More fundamentally, is there a discrete analogue of the Painlevé property?

The first potential analogue of the Painlevé property for discrete equations to appear explicitly as such in the literature is the singularity confinement property of Grammaticos, Ramani and Papageorgiou [38]. The singularity confinement test involves considering the behaviour of a sequence of iterates x_n of an equation such as (1) that goes through a singularity of the equation. In the case of equation (1), this means considering a sequence of iterates such that x_{n-1} is an arbitrary finite value and $x_n = 0$, in which case $x_{n+1} = \infty$. In order to continue the solution further it is convenient to consider the perturbed problem $x_{n-1} = k$ and $x_n = \epsilon$. After calculating any number of iterates, we then take the limit $\epsilon \to 0$. For generic equations,

the list of future iterates appears to include infinitely many singular points. However, for equation (1), the singularity appears to be confined (i.e., future iterates are finite.)

Singularity confinement is a powerful tool which has been used to discover many discrete Painlevé equations. However, Hietarinta and Viallet found an example [61] of a simple equation that appears to possess the singularity confinement property but is chaotic. They suggested that, in order to exclude such non-integrable examples, singularity confinement should be augmented by the condition that the equation has zero algebraic entropy. Zero algebraic entropy corresponds to the case in which the degree of the nth iterate of a discrete equation, considered as a rational function of the initial conditions, is bounded by a power of n [7, 29]. Veselov [126] showed that exponential growth of the degree of iterates (which is generic) corresponds to the non-existence of certain types of first integrals. Orbit dynamics over finite fields have been considered by Roberts and Vivaldi [113].

In [1], Ablowitz, Halburd and Herbst looked for a purely complex analytic analogue of the Painlevé property for discrete equations. In order to do this, discrete equations such as (1) were reinterpreted as difference equations such as

$$y(z+1) + y(z) + y(z-1) = \frac{az+b}{y(z)} + c, \tag{2}$$

for all $z \in \mathbb{C}$. A solution of equation (1) is a sequence (x_n) which, given initial conditions x_0 and x_1, can be constructed iteratively. In contrast, a solution of equation (2) is a function $y : \mathbb{C} \to \mathbb{CP}^1$. There is no unique way of extending a solution of equation (1), which 'lives' on the integers, to a solution of equation (2), which 'lives' on $\mathbb{C}$. In our analysis of the analytic properties of solutions of difference equations, we do not consider these solutions as arising from solutions of discrete equations. For ideas related to extensions to analysable functions, see Costin and Kruskal [24].

An initial value problem for equation (2) involves specifying y on, for example, the infinite strip $S = \{z : z_0 - 1 \leqslant z < z_0 + 1\}$. In general, a solution y of equation (2) that is meromorphic on S is not meromorphic on $\mathbb{C}$—it will have discontinuities on the lines $\mathrm{Re}(z) = z_0 + 2n + 1, n \in \mathbb{Z}$. The existence of meromorphic solutions of difference equations is poorly understood, yet classes of equations (such as first-order rational difference equations), including non-integrable equations, are known to admit large families of meromorphic solutions. In [1] it was suggested that a useful analogue of the Painlevé property for difference equations is the existence of sufficiently many meromorphic solutions of finite order in the sense of Nevanlinna theory.

It was observed independently by Osgood [97, 99] and Vojta [127] that there is a strong parallel between Nevanlinna theory and Diophantine approximation. The Nevanlinna approach to difference equations of Painlevé type suggests a parallel number theoretical approach to discrete equations called Diophantine integrability [46]. As well as being very easy to test numerically, this property suggests that there may be Diophantine interpretations of the discrete Painlevé equations.

This review will concentrate on using Nevanlinna theory to detect integrable difference equations. It should be stressed that this is just one of several approaches to the integrability of difference/discrete equations. There are a number of different notions of 'integrable' or 'Painlevé' in the literature, leading to slightly different classifications of equations. Yet these different approaches often agree on certain canonical examples. One of the main purposes of studying the existence of finite-order meromorphic solutions is that, as is the case for the genuine (differential) Painlevé property, this is a well-defined complex analytic property which allows us to classify and study equations without getting bogged down with the question of what exactly one means by integrability. Results such as theorem 5.4 suggest that this property singles out equations that are known to have more classical indicators of integrability such

as the existence of related linear problems, Bäcklund transformations, special solutions and relations to lattice soliton equations.

For the most part we will be concerned with meromorphic solutions of nonlinear equations. Linear equations will only be discussed in the context of recent results from Nevanlinna theory. For example, we will not describe the rich isomonodromy theory of Birkhoff [13, 14].

The outline of this review is as follows. In section 2 we will describe, by way of examples, some of the many remarkable properties of discrete Painlevé equations mentioned above, such as related linear problems and the derivation of discrete equations from the Bäcklund transformations of (differential) Painlevé equations. In section 3 we will survey a number of results on the existence of non-constant meromorphic solutions of difference equations. We outline a simple direct proof for the existence of meromorphic solutions in the first-order case. We also describe the Quispel–Roberts–Thompson (QRT) map, which is a very general class of second-order maps closely related to the discrete Painlevé equations. Much work remains to be done on the existence of meromorphic solutions in the case of general rational difference equations of order 2 or more.

Section 4 contains an introduction to basic Nevanlinna theory, in particular, we describe some simple properties of the Nevanlinna characteristic and the lemma on the logarithmic derivative. Nevanlinna theory is applied to difference equations in section 5. We describe how to find strong necessary conditions for an equation to admit a meromorphic solution of finite order. We state theorem 5.4, which describes equations of the form $w(z + 1) + w(z - 1) = R(z, w(z))$, where R is rational in w with meromorphic coefficients, which can admit solutions of finite order. We illustrate the ideas involved by considering a new example of an equation of the form $w(z + 1)w(z - 1) = R(z, w(z))$, leading to a well-known integrable difference equation often called dP_{III}. Some of the main tools used in this section are difference analogues of the lemma of the logarithmic derivative, Clunie's lemma and a value distribution result of Mohon'ko and Mohon'ko.

In section 6 we describe a number of recent results on meromorphic solutions of linear difference equations. In particular we describe a version of Wiman–Valiron theory for slow growing functions due to Ishizaki and Yanagihara, order estimates for the growth of finite-order solutions by Chiang and Feng, and a theorem concerning minimal solutions of first-order equations by Chiang and Ruijsenaars. Section 7 contains a number of results concerning the value distribution of differences of meromorphic functions. Some of these results grew out of tools developed in the classification of equations admitting finite-order meromorphic solutions. In particular we describe difference analogues of the Nevanlinna's second main theorem, Picard's theorem, defect relations and theorems concerning meromorphic functions sharing values. Other results by Bergweiler and Langley may be useful in the study of slow growth meromorphic solutions of difference equations. In section 8 we describe a q-difference analogue of the lemma on the logarithmic derivative from [4] which leads to q-difference analogues of some of the theorems mentioned above. Finally section 9 lists a number of important outstanding problems in the field.

2. Properties of discrete Painlevé equations

There are many discrete equations that are considered to be integrable discrete analogues of the Painlevé equations. Each of these equations is known to possess at least some of the following remarkable properties: the existence of a related linear (iso-monodromy) problem, the singularity confinement property, the existence of Bäcklund transformations, relations to Bäcklund transformations of differential Painlevé equations, zero algebraic entropy, Diophantine integrability. Although there is a vast literature devoted to the discrete

Painlevé equations, most of the results involve looking at only one or two properties of a small number of equations. At present there is no global theory of the discrete Painlevé equations which explains how all of these properties are related, although the algebro-geometric approach of Sakai [118], which builds on earlier work of Okamoto, might provide the starting point for such an undertaking. At the moment we cannot answer some very simple questions such as do all of the discrete Painlevé equations have Lax pairs (related linear problems). If so, how do we use these Lax pairs to describe solutions? Are all the discrete Painlevé equations related to Bäcklund transformations of other kinds of integrable equations?

This section has a very modest objective. We begin by deriving a well-known difference equation from the addition law of the Weierstrass $\wp$ function. We will then concentrate on a non-autonomous deformation of this equation, which possesses a continuum limit to the first Painlevé equation. Following Fokas, Grammaticos, and Ramani [31], we will show that this equation is the compatibility condition for a related linear problem and that it describes the Bäcklund transformations of a special case of the third Painlevé (differential) equation. The purpose of this section is to emphasize the importance from the point of view of integrable systems of the kinds of equations to which we are led in later sections by considering special complex-analytic properties.

Let ω_1 and ω_2 be two complex numbers such that $\mathrm{Im}(\omega_2/\omega_1) \neq 0$. The Weierstrass $\wp$ function is defined by

$$\wp(z) := \frac{1}{z^2} + {\sum_{m,n}}' \left\{ \frac{1}{(z - \Omega_{mn})^2} - \frac{1}{\Omega_{mn}^2} \right\}, \tag{3}$$

where $\Omega_{mn} = 2(m\omega_1 + n\omega_2)$ and ${\sum_{m,n}}'$ denotes the sum over all integer m and n excluding $(m, n) = (0, 0)$. The Weierstrass $\wp$ function is an even meromorphic function with periods $2\omega_1$ and $2\omega_2$ and satisfies the ODE

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \tag{4}$$

where

$$g_2 := 60 {\sum_{m,n}}' \Omega_{mn}^{-4} \qquad \text{and} \qquad g_3 := 140 {\sum_{m,n}}' \Omega_{mn}^{-6}.$$

The Weierstrass $\wp$ function has order 2 in the sense of Nevanlinna (see section 4.) An important property of $\wp$ is the addition law,

$$\wp(z_1 + z_2) = \frac{1}{4} \left\{ \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} \right\}^2 - \wp(z_1) - \wp(z_2). \tag{5}$$

Using this addition law, together with equation (4) and the fact that $\wp$ is even, it can be seen that, for fixed h not a pole of $\wp$, $y(z) := \wp(h(z - z_0)) - \wp(h)$ satisfies

$$y(z + 1) + y(z - 1) = \frac{\alpha y(z) + \beta}{y^2(z)}, \tag{6}$$

where $\alpha = (12\wp^2(h) - g_2)/2$ and $\beta = \wp'^2(h)$. This is a two-parameter (z_0 and g_3) family of solutions. Therefore, equation (6) admits a two-parameter family of finite-order meromorphic solutions. More generally, z_0 and g_3 can be taken to be arbitrary period 1 functions, however, care must be taken if y is required to be meromorphic.

Each of the six Painlevé equations is known to be the compatibility condition of a linear (iso-monodromy) problem of the form $\Psi_z = A(z, \zeta)\Psi$, $\Psi_\zeta = B(z, \zeta)\Psi$, for suitable choices of matrix-valued functions A and B, where z is the independent variable of the Painlevé equation and ζ is a (spectral or iso-monodromy) parameter, external to the Painlevé equation itself. A natural difference analogue of this is the system

$$\Psi(z + 1, \zeta) = U(z, \zeta)\Psi(z, \zeta), \qquad \partial_\zeta \Psi(z, \zeta) = V(z, \zeta)\Psi(z, \zeta). \tag{7}$$

In terms of the z-shift operator E_z, defined by $E_z \Psi(z, \zeta) := \Psi(z + 1, \zeta)$, the compatibility of the system (7) is $E_z \partial_\zeta \Psi(z, \zeta) = \partial_\zeta E_z \Psi(z, \zeta)$. This is equivalent to the condition

$$\partial_\zeta U(z, \zeta) = V(z + 1, \zeta) U(z, \zeta) - U(z, \zeta) V(z, \zeta). \tag{8}$$

Fokas, Grammaticos and Ramani [31] consider the particular choice

$$U(z, \zeta) := \begin{pmatrix} 2\zeta/\Omega(z+1) & -\Omega(z)/\Omega(z+1) \\ 1 & 0 \end{pmatrix}, \tag{9}$$

for some function Ω. If $\Psi(z, \zeta) = (\psi_1(z, \zeta), \psi_2(z, \zeta))^T$, then it follows from the first equation in (7) that ψ_1 satisfies

$$\Omega(z + 1)\psi_1(z + 1, \zeta) + \Omega(z)\psi_1(z - 1, \zeta) = 2\zeta\psi_1(z, \zeta),$$

which is a difference version of the Schrödinger equation. Assume that $\text{Tr}(V) = 0$. Then the compatibility condition (8) shows that V has the form

$$V(z, \zeta) = \begin{pmatrix} a(z, \zeta) & -\frac{\Omega(z)}{2\zeta}[a(z, \zeta) + a(z + 1, \zeta)] \\ \frac{\Omega(z)}{2\zeta}[a(z, \zeta) + a(z - 1, \zeta)] & -a(z, \zeta) \end{pmatrix},$$

where, on writing $a(z) \equiv a(z, \zeta)$, we have

$$\zeta[a(z + 1) - a(z)] - \Omega(z + 1)^2 \frac{a(z + 1) + a(z + 2)}{4\zeta} + \Omega(z)^2 \frac{a(z - 1) + a(z)}{4\zeta} = 1. \tag{10}$$

Seeking a solution of equation (10) of the form $a(z, \zeta) = \phi(z)/\zeta + 2\beta(z)\zeta$, gives

$$\phi(z) = \beta\Omega^2(z) + z + \alpha, \qquad \beta(z) = \beta,$$

where α and β are constants, and

$$[\phi(z - 1) + \phi(z)]\Omega(z)^2 - [\phi(z + 1) + \phi(z + 2)]\Omega(z + 1)^2 = 0. \tag{11}$$

Multiplying equation (11) by the integrating factor $\phi(z) + \phi(z + 1)$ and 'integrating' the resulting exact difference gives

$$(\phi(z) + \phi(z + 1))(\phi(z) + \phi(z - 1)) = \frac{\gamma}{\phi(z) - z - \alpha}, \tag{12}$$

where γ is an arbitrary period 1 function, which we take to be constant. Putting $y(z) = 1/(\phi(z) + \phi(z - 1))$ in equation (12) gives

$$y(z + 1) + y(z - 1) = \frac{1 - (2z + 2\alpha - 1)y(z)}{\gamma y^2(z)}, \tag{13}$$

which is a non-autonomous generalization of equation (6) with a continuum limit to the first Painlevé equation.

The following example in which a discrete Painlevé equation is constructed from Bäcklund transformations of a differential Painlevé equation is also from Fokas, Grammaticos and Ramani [31]. The third Painlevé equation is

$$w'' = \frac{w'^2}{w} - \frac{w'}{x} + \frac{1}{x}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w},$$

where α, β, γ and δ are constants. If $\gamma = 0$ but α and δ are non-zero, then we rescale x and w such that $\alpha = \delta = -1$ in the transformed equation, which takes the form

$$w'' = \frac{w'^2}{w} - \frac{w'}{x} - \frac{1}{x}(w^2 - \beta) - \frac{1}{w}. \tag{14}$$

If $w \equiv w(x, \beta)$ is a solution of equation (14) then

$$w(x; \beta + 2) = \frac{x(1 + w')}{w^2} - \frac{\beta + 1}{w}, \qquad \text{and} \tag{15}$$

$$w(x; \beta - 2) = \frac{x(1 - w')}{w^2} - \frac{\beta - 1}{w}, \tag{16}$$

are also solutions with β replaced by $\beta + 2$ and $\beta - 2$, respectively. Adding equations (15) and (16) gives

$$w(x; \beta + 2) + w(x; \beta - 2) = \frac{2x}{w^2(x; \beta)} - \frac{2\beta}{w(x; \beta)}.$$

For fixed x, let $y(z) = Aw(x; 2z + B)$, where $A \neq 0$ and B are constants. Then y satisfies

$$y(z + 1) + y(z - 1) = \frac{(az + b)y(z) + c}{y^2(z)}, \tag{17}$$

where $a = -4A^2 \neq 0$, $b = -2A^2B$ and $c = 2xA^3$, which is a well-known discrete Painlevé equation which is equivalent to equation (13).

3. Existence of meromorphic solutions

In this section we present a number of theorems on the existence of meromorphic solutions of certain classes of difference equations, some of which we will study in more detail in sections 5 and 7. For a more comprehensive account on the existence and analytic properties of solutions of difference equations, including asymptotic behaviour and formal solutions, we refer to the monographs [5, 65, 87, 88, 95, 128].

3.1. Linear difference equations

Nörlund [95] has shown that the linear difference equation

$$a_n(z)w(z + n) + \cdots + a_1(z)w(z + 1) + a_0(z)w(z) = 0 \tag{18}$$

with entire coefficients $a_0(z), \ldots, a_n(z)$ (satisfying certain growth conditions as $\text{Re}(z) \to \infty$) has a system of n meromorphic solutions on $\mathbb{C}$, linearly independent over the periodic functions. Praagman [103] applied a method due to Röhrl [114] to give an alternative proof for the existence of meromorphic solutions by considering the solutions of (18) as sections of a sheaf of modules over the sheaf of meromorphic periodic functions.

Theorem 3.1 [95, 103]. *The equation $w(z + 1) = A(z)w(z)$, with A a meromorphic invertible $n \times n$ matrix function on $\mathbb{C}$, admits a fundamental solution W, a meromorphic $n \times n$ invertible matrix function on $\mathbb{C}$.*

3.2. Nonlinear first-order difference equations

Harris and Sibuya have studied the system of first-order difference equations

$$\vec{w}(z + 1) = \vec{F}(z, \vec{w}(z)), \tag{19}$$

where $\vec{w}$ and $\vec{F}$ are n-dimensional vectors [51–53]. They showed that if $\vec{F}(\infty, \vec{0}) = \vec{0}$ and the components of $\vec{F}$ are holomorphic in a certain region, then equation (19) has a holomorphic solution in a suitable angular domain. Although their method works for large classes of difference equations, proving the global existence of non-trivial meromorphic solutions of difference equations in the complex plane requires a different approach. Fatou [30], and later

Shimomura proved the existence of entire solutions of a large class of first-order difference equations.

Theorem 3.2 (Shimomura [121]). *For any non-constant polynomial $P(w)$, the difference equation*

$$w(z + 1) = P(w) \tag{20}$$

has a non-trivial entire solution.

Some of the properties of these solutions are described in [3].

The case where the right side of (20) is rational in w has been split into several different subcases, which have been treated by Kimura [78], Hirai [63] and Yanagihara [129]. The statement of the following theorem is due to Yanagihara [129].

Theorem 3.3 [63, 78, 129]. *For any non-constant rational function $R(w)$, the difference equation*

$$w(z + 1) = R(w) \tag{21}$$

has a non-trivial meromorphic solution.

Some of the properties of these solutions are described in [2].

We outline here a straightforward proof of the existence of meromorphic solutions of (21) based on the methods used in [78, 121, 129] and on the Banach fixed point theorem. Full details of this approach can be found in [49].

In the Riccati or linear case $\deg_w(R) = 1$ equation (21) is explicitly solvable in terms of exponential functions, see, for instance [82]. If $\deg_w(R) \geqslant 2$, then by [75] R has a fixed point γ such that either

$$|R'(\gamma)| > 1 \tag{22}$$

or

$$R'(\gamma) = 1. \tag{23}$$

In the case (22) equation (21) can be mapped into a Schröder functional equation, which is known to have meromorphic solutions (see, e.g., [121, 129]). We choose a slightly different approach which is based on a direct application of the Banach fixed point theorem. We will now show that (21) has a unique meromorphic solution in the complex plane such that

$$(w(z) - \gamma)\lambda^{-z} \longrightarrow \alpha, \quad \text{as} \quad \text{Re}(z) \longrightarrow -\infty \tag{24}$$

where $|R'(\gamma)| =: \lambda > 1$ and $\alpha \in \mathbb{C}$.

Expanding the right side of (21) as a Maclaurin series, and applying the transformation $w(z) \to \lambda^z w(z) + \gamma$ takes equation (21) into the form

$$w(z + 1) - w(z) = \sum_{j=2}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)z-1} w(z)^j, \tag{25}$$

which implies

$$w(z) - w(z - m) = \sum_{j=2}^{\infty} \sum_{k=1}^{m} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)(z-k)-1} w(z - k)^j.$$

Letting $m \longrightarrow \infty$, assumption (24) yields the formal identity

$$w(z) = \alpha + \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)(z-k)-1} w(z - k)^j, \tag{26}$$

which suggests to define an operator T by

$$T[w](z) = \alpha + \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)(z-k)-1} w(z-k)^j. \tag{27}$$

Let X be the set of all functions $g(z)$, analytic and bounded in

$$D(s, t) = \{z : \mathrm{Re}(z) < -s, \ \mathrm{Im}(z) > -t, \ t > 0\} \tag{28}$$

for which $g(z) \longrightarrow \alpha$ as $\mathrm{Re}(z) \longrightarrow -\infty$ and $\|g - \alpha\| \leqslant b$, where

$$\|g - \alpha\| = \sup_{z \in D(s,t)} |g(z) - \alpha|. \tag{29}$$

Then, choosing s sufficiently large, T is a contraction mapping in the Banach space X. Banach's fixed point theorem implies that T has a unique fixed point $g(z)$ in X. This fixed point is the sought after solution of (21) satisfying (24), analytic in $D(s, t)$. By analytic continuation the solution $g(z)$ is meromorphic in $D(+\infty, t)$. Since t is arbitrary, $g(z)$ is in fact meromorphic in the whole complex plane.

The case (23) can be found in a paper due to Kimura [78]. He uses a fixed point theorem due to Hukuhara based on the theory of normal families [22, 120] to study a general problem related to the iteration of analytic functions. The existence of meromorphic solutions of (21) in case (23) follows as a special case from his results. Since we are only concerned with the existence of meromorphic solutions of difference equations, for our purposes it is sufficient to apply the Banach fixed point theorem in case (23). We will show that if $\alpha \in \mathbb{C}$, $\delta \in (0, 1)$ and $m \in \mathbb{N}$ is the smallest number such that $R^{(m+1)}(\gamma) \neq 0$, then there exists a constant $s > 0$, and a unique solution $w(z)$ of (21), meromorphic in the complex plane, such that for all $z \in D(s) := \{z : \mathrm{Re}(z) < -s\}$,

$$w(z) = \gamma - \frac{C_1}{z + \alpha + \beta \log z + W(z)}$$

if $m = 1$, and

$$w(z) = \gamma - \frac{C_m}{\left(z + \alpha + \beta z^{\frac{m-1}{m}} + W(z) \right)^{\frac{1}{m}}},$$

if $m \geqslant 2$. Here β is a fixed constant,

$$C_m = \left(\frac{m}{(m+1)!} R^{(m+1)}(\gamma) \right)^{-\frac{1}{m}}$$

for $m \in \mathbb{N}$, and $|W(z)| \leqslant |z|^{-\frac{1}{m}+\delta}$ for all $z \in D(s)$.

By expanding the right side of (21) as a Taylor series and by substituting $w(z) \to 1/w(z) + \gamma$, we obtain

$$w(z+1) = \frac{w(z)}{1 + \sum_{j=2}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) w(z)^{-j+1}} = w(z) \left(1 - \frac{1}{2} R''(\gamma) w(z)^{-1} + \cdots \right). \tag{30}$$

From now on we assume that $R''(\gamma) \neq 0$. The details in the case $R''(\gamma) = 0$ can be found in [49]. By denoting $h(z) = -2w(z)/R''(\gamma)$ equation (30) takes the form

$$h(z+1) = h(z) + 1 + \sum_{j=1}^{\infty} c_j h(z)^{-j},$$

where $c_j \in \mathbb{C}$ for all $j \in \mathbb{N}$. In summary, we have transformed equation (21) into

$$w(z+1) = F(w(z)), \tag{31}$$

where

$$F(z) = z + 1 + \sum_{j=1}^{\infty} c_j z^{-j}. \tag{32}$$

Let X be the family of analytic functions $W(z)$ such that

$$|W(z)| \leqslant |z|^{-1+\delta} \tag{33}$$

for all $z \in D(s)$. Then X is a Banach space with respect to the sup-norm. By a straightforward calculation it can be seen that the operator

$$
\begin{aligned}
T[W](z) = {} & \sum_{k=1}^{\infty} \frac{c_1}{z-k} \sum_{n=1}^{\infty} \left(-\frac{\alpha + \beta \log(z-k) + W(z-k)}{z-k} \right)^n \\
& + \sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \frac{c_j}{(z-k+\alpha+\beta \log(z-k)+W(z-k))^j} \\
& + \beta \sum_{k=1}^{\infty} \sum_{j=2}^{\infty} (-1)^j \frac{1}{j(z-k)^j},
\end{aligned}
\tag{34}
$$

where $\beta = c_1$, is a contraction mapping satisfying

$$|T[W](z)| \leqslant |z|^{-1+\delta} \tag{35}$$

for all $z \in D(s)$. Therefore, the right side of (34) is absolutely and uniformly convergent, and hence $T[W](z)$ is analytic in $D(s)$. Thus, by Banach's fixed point theorem, the mapping $T : X \longrightarrow X$ has a unique fixed point. This is equivalent with the fact that

$$Y(z) = z + \alpha + \beta \log z + W(z)$$

is a solution of (31), by our choice of $T[W]$. The final step is to continue the analytic solution $Y(z)$ into a meromorphic solution in the whole complex plane by using equation (21).

3.3. The QRT map

Quispel, Roberts and Thompson have introduced a general system of second-order mappings which is explicitly integrable in terms of elliptic functions [105, 106]. Many discrete Painlevé equations have been first derived by implementing a discrete integrability criterion to a non-autonomous form of a mapping within the QRT family [109]. The QRT map is a generalization of the McMillan map [86].

The QRT family is defined by the system of two equations

$$
\begin{aligned}
x_{n+1} &= \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)} \\
y_{n+1} &= \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}
\end{aligned}
\tag{36}
$$

where

$$
\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \delta_0 & \varepsilon_0 & \zeta_0 \\ \kappa_0 & \lambda_0 & \mu_0 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \delta_1 & \varepsilon_1 & \zeta_1 \\ \kappa_1 & \lambda_1 & \mu_1 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix},
$$

$$
\begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix} = \begin{pmatrix} \alpha_0 & \delta_0 & \kappa_0 \\ \beta_0 & \varepsilon_0 & \lambda_0 \\ \gamma_0 & \zeta_0 & \mu_0 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times \begin{pmatrix} \alpha_1 & \delta_1 & \kappa_1 \\ \beta_1 & \varepsilon_1 & \lambda_1 \\ \gamma_1 & \zeta_1 & \mu_1 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}
$$

and the coefficient matrices are constants. In the symmetric case

$$\begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \varepsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix} = \begin{pmatrix} \alpha_i & \delta_i & \kappa_i \\ \beta_i & \varepsilon_i & \lambda_i \\ \gamma_i & \zeta_i & \mu_i \end{pmatrix}, \qquad i = 0, 1,$$

the QRT family reduces into a single equation

$$w_{n+1} = \frac{f_1(w_n) - w_{n-1} f_2(w_n)}{f_2(w_n) - w_{n-1} f_3(w_n)} \tag{37}$$

by taking $x_n = w_{2n}$ and $y_n = w_{2n+1}$ in (36).

We will now follow a well-known procedure, see e.g. [6, 107], to show that the symmetric QRT difference equation

$$\overline{w} = \frac{f_1(w) - \underline{w} f_2(w)}{f_2(w) - \underline{w} f_3(w)}, \tag{38}$$

related to the discrete integrable equation (37), possesses generically a two-parameter family of finite-order meromorphic solutions. (Here we have suppressed the z-dependence of $w(z)$ by writing $w \equiv w(z)$, $\overline{w} \equiv w(z+1)$ and $\underline{w} \equiv w(z-1)$.) For the integration of the full asymmetric system (36) we refer to [64, 107]. Equation (38) can be integrated to obtain

$$\overline{w}^2 w^2 + (\beta_0 + \beta_1 C_1)(\overline{w}^2 w + \overline{w} w^2) + (\gamma_0 + \gamma_1 C_1)(\overline{w}^2 + w^2)$$

$$+ (\zeta_0 + \zeta_1 C_1)(\overline{w} + w) + (\varepsilon_0 + \varepsilon_1 C_1)\overline{w}w + (\mu_0 + \mu_1 C_1) = 0, \tag{39}$$

where C_1 is an arbitrary periodic function chosen here to be constant. Apart from certain degenerate special cases, the Möbius transformation

$$w \longrightarrow \frac{aw + b}{cw + d},$$

where a, b, c, d are suitable constants, takes equation (39) into the form

$$\overline{w}^2 w^2 + A(\overline{w}^2 + w^2) + 2B\overline{w}w + 1 = 0, \tag{40}$$

where the constants A and B depend on a, b, c, d and C_1, and on the coefficients of (39). From now on we assume that neither of the coefficients A or B vanish concentrating thus only on the generic case. Keeping this in mind we define the parameters k and η by the equations

$$A = -\frac{1}{k \operatorname{sn}^2 \eta},$$

and

$$B = \frac{\operatorname{cn} \eta \operatorname{dn} \eta}{k \operatorname{sn}^2 \eta}.$$

These choices of A and B imply that

$$k + k^{-1} = (B^2 - A^2 - 1)A^{-1}.$$

Therefore, considering equation (40) as a quadratic equation for $\overline{w}$, and using the transformation

$$w = k^{\frac{1}{2}} \operatorname{sn} u,$$

where $\operatorname{sn} u$ denotes the Jacobian elliptic sn function with argument u and modulus k, we have

$$\operatorname{sn} \overline{u} = \frac{\operatorname{cn} \eta \operatorname{dn} \eta \operatorname{sn} u \pm \operatorname{sn} \eta \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 \eta \operatorname{sn}^2 u}.$$

This is solved by $u = \eta z + C_2$, where $C_2 \in \mathbb{C}$ is a free parameter. Summarizing the above reasoning,

$$w = \frac{ak^{\frac{1}{2}} \operatorname{sn}(\eta z + C_2) + b}{ck^{\frac{1}{2}} \operatorname{sn}(\eta z + C_2) + d} \tag{41}$$

is a meromorphic solution of (38), where $C_2 \in \mathbb{C}$, and a, b, c, d, η and k depend on the coefficients of (38) and on the free parameter $C_1 \in \mathbb{C}$.

3.4. Nonlinear higher order difference equations

We conclude this section with an existence theorem due to Yanagihara on a class of higher order difference equations.

Theorem 3.4 (Yanagihara [131]). *For any rational function*

$$R(w) = \frac{a_p w^p + \cdots + a_0}{b_q w^q + \cdots + b_0}, \tag{42}$$

where $a_p, \ldots, a_0 \in \mathbb{C}, b_q, \ldots, b_0 \in \mathbb{C}$ and $p \geqslant q + 2$, the difference equation

$$\alpha_n w(z + n) + \cdots + \alpha_1 w(z + 1) = R(w), \qquad \alpha_n, \ldots, \alpha_1 \in \mathbb{C},$$

has a non-trivial meromorphic solution.

The method of proof of theorem 3.4 is similar to that of theorem 3.3. A fixed point theorem is first applied to find an analytic solution of (42) in a suitable sector, which is then continued to a meromorphic solution in the whole complex plane. Although this method works for large classes of difference equations, all meromorphic solutions found with it are free from poles in large parts of the complex plane. Therefore the method cannot be applied to equations solvable, for instance, in terms of elliptic functions, such as the QRT system in section 3.3.

4. Nevanlinna theory

We have seen that large classes of difference equations admit meromorphic solutions. Therefore the existence of meromorphic solutions alone is not a good indicator of integrability. In order to characterize well-behaved meromorphic solutions we turn to Nevanlinna theory [92].

Nevanlinna theory has applications and analogies in many different fields of mathematics, such as differential equations [27, 34, 62, 40, 79], difference equations [121, 128, 129] number theory [98, 99, 115, 127], Brownian motion [18] and even mathematical logic [60]. Recently, there has been increasing interest in applying Nevanlinna theory to study meromorphic solutions of complex difference equations [20, 21, 48, 58, 67, 80], and in particular, to detect integrability in discrete equations [1, 49, 50, 111].

In what follows we will only give a brief outline of classical Nevanlinna theory without presenting proofs of the theorems. A more comprehensive description of Nevanlinna theory can be found, for example, in [19, 34, 54, 72]. For excellent historical accounts on the development of the theory we refer to [57, 81].

4.1. Basic theory

Picard's classical theorem states that a non-constant entire function can only omit one value in the complex plane [102]. This result was generalized by Borel who was able to go much deeper in the study of value distribution of entire functions [16]. He considered entire functions with finite *order of growth*

$$\sigma(f) := \limsup_{r \longrightarrow \infty} \frac{\log \log M(r, f)}{\log r}, \tag{43}$$

where $M(r, f) := \max_{|z|=r} |f(z)|$ is the maximum modulus of $f(z)$ in a disc of radius r. Borel showed that the number of a-points, i.e. the number of roots of the equation $f(z) = a$, of a non-constant entire function of finite order is determined by the rate of growth of the function, with at most one exceptional value a. Borel's result was later extended to infinite order functions by Blumenthal [15].

Picard's theorem immediately generalizes to the meromorphic case. Namely, if $f(z)$ is a meromorphic function omitting three distinct values a, b and c, then $1/(f(z) - a)$ is an entire function omitting $1/(b - a)$ and $1/(c - a)$, which implies that $f(z)$ must be a constant. Generalizing Borel's theorem to meromorphic functions is much more complicated. For one thing, it is not even immediately obvious how the notion of order should be defined for meromorphic functions. The first satisfactory extension of Borel's result to meromorphic functions was given by Nevanlinna [92]. He approached the problem by associating to each meromorphic function $f(z)$ three real-valued functions $m(r, f)$, $N(r, f)$ and $T(r, f)$ which could be used indirectly to obtain information on the growth and value distribution of the meromorphic function $f(z)$ more efficiently than what had so far been achieved by a direct analysis of the meromorphic function itself.

The *proximity function* $m(r, f)$ is defined as the integral

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(r \, e^{i\theta}) \right| d\theta,$$

where $\log^+ x := \max\{0, \log x\}$ for all $x > 0$. The proximity function describes, roughly speaking, how close on the average the values of $f(z)$ are to infinity on the circle $|z| = r$. Similarly, the growth of the function $m(r, a) := m\left(r, \frac{1}{f-a}\right)$ correlates with the proximity of the values of $f(z)$ to the value $a \in \mathbb{C}$: the closer the values of $f(z)$ are to a on the circle $|z| = r$, the larger the function $m(r, a)$ is.

The *counting function* $N(r, f)$ is an integral of the number of poles of $f(z)$ inside the disc $|z| \leqslant r$ with respect to the logarithmic measure dr/r. It is sometimes also referred to as the *integrated counting function*, and it is defined as

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r,$$

where $n(r, f)$ is the number of poles of $f(z)$, counting multiplicities, in the disc $|z| \leqslant r$. Similarly, $N(r, a) := N\left(r, \frac{1}{f-a}\right)$ counts the number of a-points of $f(z)$ (i.e. the number of points $z \in \mathbb{C}$ such that $f(z) = a$) in the disc of radius r centred at the origin.

The sum of the proximity function and the counting function is denoted by

$$T(r, f) := m(r, f) + N(r, f)$$

and it is called as the *characteristic function* of $f(z)$. The growth of the function $T(r, f)$ is a good measure of the complexity of the meromorphic function $f(z)$. The Nevanlinna characteristic is a convex increasing function of $\log r$, and differentiable almost everywhere. Moreover, by defining the order of growth of a meromorphic function $f(z)$ as

$$\rho(f) := \limsup_{r \longrightarrow \infty} \frac{\log T(r, f)}{\log r},$$

it follows that $\rho(f) = \sigma(f)$ whenever $f(z)$ in entire [54]. Another remarkable property which the characteristic function possesses is known as the first main theorem of Nevanlinna theory,

$$T(r, f) = T\left(r, \frac{1}{f - a}\right) + O(1). \tag{44}$$

The first main theorem implies that if $f(z)$ takes a value $a \in \mathbb{C} \cup \{\infty\}$ fewer times than average, i.e., the counting function $N(r, a)$ is relatively small, then the proximity function $m(r, a)$ must be large, and vice versa. Loosely speaking, if a meromorphic function assumes a certain value a relatively few times, the values of $f(z)$ are 'near' the value a in a large part of the complex plane.

The second main theorem of Nevanlinna theory extends Borel's result on the value distribution of entire functions to meromorphic functions. The exact statement and further discussion may be found in section 7.2 below.

4.2. Exceptional sets

In Nevanlinna theory one cannot usually avoid dealing with equations and inequalities which hold for 'most' values in the positive real axis. The 'small' set in which the desired property is not valid is called an *exceptional set*. When talking about an exceptional set one must always specify how large the set in question is. For example, if a set $E \subset [0, \infty)$ satisfies $\int_E \, dt < \infty$ it is said that E has *finite linear measure*.

In this review we mostly end up dealing with slightly larger exceptional sets which may be of infinite linear measure, but have *finite logarithmic measure* $\int_E \frac{1}{t} \, dt < \infty$. For instance, if the characteristic function of a meromorphic function, say $g(z)$, satisfies $T(r, g) = o(T(r, f))$ outside of an exceptional set of r-values with finite logarithmic measure then it is said that $g(z)$ is *small compared to* $f(z)$, and the notation

$$T(r, g) = S(r, f) \tag{45}$$

is used. Intuitively speaking (45) just means that $T(r, g)$ is very small compared to $T(r, f)$ for most $r \geqslant 0$. We also denote by $\mathcal{S}(f)$ the family of all meromorphic functions which are small compared to $f(z)$.

4.3. Useful properties of the characteristic function $T(r, f)$

The characteristic function $T(r, f)$ possesses a number of useful properties which make it a relatively easy function to work with. As we already pointed out, the characteristic function is a convex increasing function of $\log r$, differentiable with respect to r almost everywhere. In addition to this, the characteristic function provides a very convenient way to characterize rational functions. Namely, a meromorphic function $g(z)$ is rational if and only if $T(r, g) = O(\log r)$. This property is particularly handy when one is analysing the growth of non-rational meromorphic solutions of differential and difference equations with rational coefficients: the growth of the slower growing rational coefficients is relatively easy to distinguish from the growth of the fast growing meromorphic solution.

In addition to the first and second main theorems of Nevanlinna theory, there are a number of identities and inequalities for the characteristic function, counting function and the proximity function which are used frequently in Nevanlinna theory. Let $f(z)$, $g(z)$ and $h(z)$ be meromorphic functions, and let $n \in \mathbb{N}$. Then

$$T(r, f + g) \leqslant T(r, f) + T(r, g) + O(1) \tag{46}$$

$$T(r, fg) \leqslant T(r, f) + T(r, g) \tag{47}$$

$$T(r, f^n) = nT(r, f) \tag{48}$$

$$T(r, fg + gh + hf) \leqslant T(r, f) + T(r, g) + T(r, h) + O(1) \tag{49}$$

$$T(r+1, f) = T(r, f) + S(r, f) \qquad \text{if } f \text{ is of finite order.} \tag{50}$$

Relations (46)–(49) hold also for the counting function $N(r, f)$ and the proximity function $m(r, f)$, while the inequality (50) is true for the counting function but not necessarily for the proximity function. For a proof of the relations (46)–(48) see any standard reference in the Nevanlinna theory, for instance [54], for (49) see [111], and for (50) see [50]. In addition to (46)–(50) the identity

$$m\left(r, \frac{f'}{f}\right) = S(r, f) \tag{51}$$

is very important in Nevanlinna theory. Relation (51) is called the lemma on the logarithmic derivative and it is an essential part of the proof of the second main theorem (110).

Finally, an identity originally due to Valiron [124] and generalized by Mohon'ko [89], has proved to be an extremely useful tool in the study of meromorphic solutions of differential, difference and functional equations. If $R(z, f)$ is a rational function of $f(z)$ with meromorphic coefficients a_λ such that $T(r, a_\lambda) = S(r, f)$ for all λ, then

$$T(r, R(z, f)) = \deg_f(R(z, f))T(r, f) + S(r, f). \tag{52}$$

In particular, if all coefficients of $R(z, f)$ are rational functions and $f(z)$ is non-rational, then $S(r, f)$ may be replaced with an error term of the growth $O(\log r)$ without an exceptional set.

Let us demonstrate the power of Nevanlinna theory in the analysis of meromorphic solutions of differential equations with a concrete example. Let $w(z)$ be a non-rational meromorphic solution of the differential equation

$$w' = P(z, w), \tag{53}$$

where $P(z, w)$ is a polynomial in $w(z)$ with rational coefficients. Then, by (52), we have

$$T(r, w') = \deg_w(P(z, w))T(r, w) + O(\log r). \tag{54}$$

By the lemma on the logarithmic derivative (51) and (47), we obtain

$$T(r, w') \leqslant N(r, w') + m(r, w) + m\left(r, \frac{w'}{w}\right) \leqslant 2T(r, w) + O(\log r). \tag{55}$$

Now, since $w(z)$ is non-rational $T(r, w) \neq O(\log r)$, and so by combining (54) and (55) we have $\deg_w(P(z, w)) \leqslant 2$. We conclude that if equation (53) has at least one non-rational meromorphic solution, then (53) must reduce to a Riccati differential equation

$$w' = a_2(z)w^2 + a_1(z)w + a_0(z)$$

with rational coefficients. We have just reproduced Yosida's proof [132] of the classical Malmquist theorem [85] in a special case. The treatment of the full rational case by using Nevanlinna theory is not much more complicated, see for example [79]. See also [27].

5. Difference equations admitting finite-order meromorphic solutions

In this section we explore the existence of sufficiently many solutions as a detector of integrability.

5.1. Difference equations of Painlevé type

Yanagihara studied meromorphic solutions of nonlinear difference equations by using methods from Nevanlinna theory [129]. The following theorem can be understood as a difference analogue of Malmquist's theorem.

Theorem 5.1 (Yanagihara [129]). *If the first-order difference equation*

$$\overline{w} = R(z, w), \tag{56}$$

where $R(z, w)$ is rational in both arguments, admits a non-rational meromorphic solution of finite order, then $\deg_w(R) = 1$.

Equation (56) with $\deg_w(R) = 1$ is the difference Riccati equation,

$$w(z + 1) = \frac{\alpha(z)w(z) + \beta(z)}{\gamma(z)w(z) + \delta(z)}, \tag{57}$$

which possesses a continuum limit to the differential Riccati equation and is linearizable [110]. If $\gamma \not\equiv 0$ then equation (57) is linearized by the transformation

$$w(z) = \frac{\alpha(z-1)}{\gamma(z-1)} \left[\frac{u(z) - u(z-1)}{u(z)} \right],$$

where u satisfies the second-order linear difference equation

$$p(z)u(z+1) + q(z)u(z) + r(z)u(z-1) = 0,$$

where

$$p(z) = \gamma(z-1)[\alpha(z)\delta(z) - \beta(z)\gamma(z)],$$
$$q(z) = -\alpha(z)[\alpha(z-1)\gamma(z) + \gamma(z-1)\delta(z)],$$
$$r(z) = \alpha(z-1)\alpha(z)\gamma(z).$$

Yanagihara's work has prompted a number of generalizations of theorem 5.1 to higher order difference equations [1, 58, 80, 111, 130]. We will now have a closer look at some of these results connected to difference Painlevé equations. Difference equations of the type

$$\overline{w} \star \underline{w} = R(z, w), \tag{58}$$

where $R(z, w)$ is rational in both of its arguments, and the operation $\star$ stands either for the addition or the multiplication, were studied in [1].

Theorem 5.2 (Ablowitz, Halburd and Herbst [1]). *If the second-order difference equation (58) admits a non-rational meromorphic solution of finite order, then $\deg_w(R) \leqslant 2$.*

The class of equations (58) with $\deg_w(R) \leqslant 2$ contains many equations considered to be of Painlevé type, including equations known as the difference Painlevé I–III. The following theorem is due to Ramani, Grammaticos, Tamizhmani and Tamizhmani [111].

Theorem 5.3 (Ramani, Grammaticos, Tamizhmani and Tamizhmani [111]). *If the second-order difference equation*

$$(\overline{w} + w)(w + \underline{w}) = \frac{P(z, w)}{Q(z, w)}, \tag{59}$$

where $P(z, w)$ and $Q(z, w)$ are polynomials in w having rational coefficients and no common roots, admits a non-rational meromorphic solution of finite order, then $\deg_w(P) \leqslant 4$ and $\deg_w(Q) \leqslant 2$.

The class of equations singled out in theorem 5.3 contains the difference Painlevé IV equation, first derived by the singularity confinement method in [109]. Ramani *et al* have also given an analogue of theorem 5.3 which singles out a class of equations containing the difference Painlevé V equation [111].

The fact that many equations widely considered to be integrable lie within the classes of equations (58) with $\deg_w(R) \leqslant 2$, and (59) with $\deg_w(P) \leqslant 4$ and $\deg_w(Q) \leqslant 2$, supports the assertion that the existence of finite-order meromorphic solutions is a good detector of discrete integrable equations. On the other hand, many equations within these classes are generally considered to be non-integrable. In theorems 5.4 and 5.5 we have shown that equation (58) admits finite-order meromorphic solutions only in some very special cases which correspond to discrete equations considered to be integrable.

Instead of looking at equations with just rational coefficients, we have chosen a more general starting point, since some of the coefficients of discrete Painlevé equations are non-rational and we wish to single them out in their full generality. We say that a meromorphic solution w of a difference equation is *admissible* if all coefficients of the equation are in $\mathcal{S}(w)$ (see [79] for admissible solutions in differential equations.) This means that the solution has faster growth than any of the coefficients in the sense of Nevanlinna theory (recall the exact definition from section 4.2.) For instance, all non-rational meromorphic solutions of an equation with rational coefficients are admissible. Note that theorems 5.2 and 5.3 remain true if 'rational' is replaced by 'meromorphic' and 'non-rational' is replaced by 'admissible'.

Theorem 5.4 [50]. *If the equation*

$$\overline{w} + \underline{w} = R(z, w), \tag{60}$$

where $R(z, w)$ is rational and irreducible in w and meromorphic in z, has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation

$$\overline{w} = \frac{\overline{p}\, w + q}{w + p}, \tag{61}$$

where $p, q \in \mathcal{S}(w)$, or equation (60) can be transformed by a linear change in w to one of the following equations:

$$\overline{w} + w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + \kappa_1 \tag{62}$$

$$\overline{w} - w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + (-1)^z \kappa_1 \tag{63}$$

$$\overline{w} + \underline{w} = \frac{\pi_1 z + \kappa_1}{w} + \frac{\pi_2}{w^2} \tag{64}$$

$$\overline{w} + \underline{w} = \frac{\pi_1 z + \pi_3}{w} + \pi_2 \tag{65}$$

$$\overline{w} + \underline{w} = \frac{(\pi_1 z + \kappa_1)w + \pi_2}{(-1)^{-z} - w^2} \tag{66}$$

$$\overline{w} + \underline{w} = \frac{(\pi_1 z + \kappa_1)w + \pi_2}{1 - w^2} \tag{67}$$

$$\overline{w}w + w\underline{w} = p \tag{68}$$

$$\overline{w} + \underline{w} = p\, w + q \tag{69}$$

where $\pi_k, \kappa_k \in \mathcal{S}(w)$ are arbitrary finite-order periodic functions with period k.

Equation (62) arises from the theory of orthogonal polynomials (see e.g. [83, 122]). It also appears in the matrix model approach to two-dimensional quantum gravity [12, 17, 32]. Historically, equation (62) is known as the discrete Painlevé I equation. The choice of name stems from the fact that a continuous limit, such as $w = -1/2 + \varepsilon^2 u$, $\kappa_1 = -3$, $\pi_1 z + \pi_2 = -(3+2\varepsilon^4 t)/4$, $\varepsilon \to 0$, may be used to map (62) to the Painlevé I equation $u'' = 6u^2 + t$ [31]. In addition, equation (62) possesses a Lax pair, and it may be integrated by using isomonodromy techniques [32, 100]. Equation (64) is a known alternate difference Painlevé I equation [31, 37, 108]. It can also be mapped to the continuous Painlevé I equation by a suitable continuous limit, and its Lax pair has been given in [31]. Equation (65) is a known integrable equation with continuum limits to Painlevé I and IV, and its Lax pair has been given in [36, 39].

Equation (67) was found in connection with unitary matrix models of two-dimensional quantum gravity [101], and it was identified as the difference Painlevé II based on a continuum limit to the continuous Painlevé II equation. Equation (67) was also obtained as a similarity reduction of the discrete mKdV equation [94]. It possesses many special properties, including Lax pairs [74, 100], special Airy-type solutions [77, 119] and discrete Miura and auto-Bäcklund transformations [37].

Extensive studies of equations (62)–(65) and (67) suggest that they are all integrable [35]. In addition to possessing many properties indicative of integrability, including Lax pairs, they pass the singularity confinement test and have zero algebraic entropy [38, 61, 96]. They are also a part of the coalescence cascade for the discrete Painlevé equations [37]. Equation (66) is a slight variation of (67), and it appears to be related to an integrable alternating mapping introduced in [104]. Equation (61) is a difference Riccati equation, and (69) a linear difference equation. Equation (68) is linear in $w\underline{w}$ and possesses finite-order meromorphic solutions of many choices of p. Although some of the equations in theorem 5.4 can be transformed to each other, it is not clear whether such transformations preserve the admissibility of the solution w. It should also be noted that it is possible that two discrete equations can be transformed into each other while the corresponding difference equations cannot. This often comes down to the fact that, for $n \in \mathbb{Z}$ we have identities such as $((-1)^n)^2 = 1$, but for $z \in \mathbb{C}$, $((-1)^z)^2 = \exp(2\pi \mathrm{i} z)$. We conclude that the list of equations (61)–(69) contains all known integrable equations of the form (60) and apparently no non-integrable equations.

Instead of just repeating the proof of theorem 5.4 here, we take the opportunity to apply the method used to prove theorem 5.4 in [50] to single out the difference Painlevé III equation out of a natural class of difference equations. To this end, assume that w is an admissible meromorphic solution of (58). By combining the identity (52) with the fact that

$$T(r, w(z \pm 1)) \leqslant (1 + \varepsilon)T(r + 1, w) + O(1)$$

holds for $\varepsilon > 0$ when r is sufficiently large [129] we obtain

$$T(r + 1, w) \geqslant \frac{\deg_w(R)}{2(1 + \varepsilon)} T(r, w) + S(r, w). \tag{70}$$

Inequality (70) was obtained by following the reasoning in [1, 129], the only difference being that here the exceptional set associated with the error term $S(r, w)$ is bigger. If the degree of $R(z, w)$ with respect to w is three or larger, then there exists an $\alpha < 1$ such that

$$T(r, w) \leqslant \alpha T(r + 1, w) \tag{71}$$

outside of a possible exceptional set E with finite logarithmic measure. If w is of finite order, then we arrive at a contradiction by combining (50) and (71). We conclude that if (58) has a finite-order meromorphic solution then $\deg_w(R) \leqslant 2$.

In the following theorem we consider a natural subclass of (58) with $\deg_w(R(z, w)) \leqslant 2$, and show that the existence of at least one finite-order meromorphic solution w is sufficient to single out the difference Painlevé III equation from this subclass, provided that w is not also an admissible solution of a difference Riccati equation. The full classification of (58) will be done elsewhere.

Theorem 5.5. *Let w be an admissible finite-order meromorphic solution of the equation*

$$\overline{w}\underline{w} = \frac{c_2(w - c_+)(w - c_-)}{(w - a_+)(w - a_-)} =: R(z, w), \tag{72}$$

where the coefficients are meromorphic functions, $c_2 \not\equiv 0$ and $\deg_w(R) = 2$. If the order of the poles of w is bounded, then either w satisfies a difference Riccati equation

$$\overline{w} = \frac{pw + q}{w + s}, \tag{73}$$

where $p, q, s \in \mathcal{S}(w)$, or equation (72) can be transformed by a bilinear change in w to one of the equations

$$\overline{w}\underline{w} = \frac{\gamma w^2 + \delta \lambda^z w + \gamma \mu \lambda^{2z}}{(w - 1)(w - \gamma)}, \tag{74}$$

$$\overline{w}\underline{w} = \frac{w^2 + \delta \, e^{i\pi z/2} \lambda^z w + \mu \lambda^{2z}}{w^2 - 1}, \tag{75}$$

where $\lambda \in \mathbb{C}$, and $\delta, \mu, \gamma \in \mathcal{S}(w)$ are arbitrary finite-order periodic functions such that δ and γ have period 2 and μ has period 1.

Equation (74) is the difference Painlevé III equation, first identified by the singularity confinement method [109]. It possesses a Lax pair [100], special discrete Riccati solutions [76] and Schlesinger transforms [73].

Compared to theorem 5.4, the assertion of theorem 5.5 contains an additional assumption concerning the orders of the poles of $w(z)$. This assumption can in fact be removed, and it is added here to avoid a number of technicalities which appear in the special case where the multiplicities of the poles of w are allowed to grow without limit. In this case, for instance, it could happen that whenever w has a pole, one of the coefficients of (72) also has a pole. Then, if the multiplicities of the poles of the coefficients remain small compared to the multiplicities of the poles of w, it is still possible that w is admissible. The difficulties caused by this fact can be overcome, but presenting all the details here would not serve our purpose. Instead, we assume in what follows that the multiplicities of the poles of w remain bounded and refer to [50] for a method of how to deal with sequences of poles where the multiplicities grow without limit.

5.2. Tools from Nevanlinna theory

The proof of theorem 5.4 in [50] and the proof of theorem 5.5 rely on some recent results on Nevanlinna theory for the shift operator $\Delta_c f := f(z + c) - f(z)$. We state these results here before proceeding to prove theorem 5.5.

The lemma on the logarithmic derivative is the main ingredient in the proof of the second main theorem of Nevanlinna theory. It has also proved to be extremely useful in the analysis of value distribution of meromorphic solutions of differential equations [79]. The first result in this subsection is a difference analogue of the lemma on the logarithmic derivative and it is the foundation on which the Nevanlinna theory for the shift operator is built [47, 48].

Chiang and Feng [20] obtained almost the same result as theorem 5.6 below, independently of [48], in a study concerning finite-order meromorphic solutions of linear difference equations.

Theorem 5.6 [47, 48]. *Let $f(z)$ be a meromorphic function of finite order and let $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f).$$

In this section $S(r, f)$ represents any function of r which is $o(T(r, f))$ for all r outside of an exceptional set with finite logarithmic measure. Stronger estimates of this error term for the theorems presented in this section are given in [47, 48].

Theorem 5.6 combined with standard methods from the Nevanlinna theory of differential equations is all that is needed to prove some rather powerful results on the value distribution of finite-order meromorphic solutions of large classes of nonlinear difference equations of arbitrarily high order. We state these results in their full generality first and then apply them to study finite-order meromorphic solutions of difference Painlevé equations.

Let c_j, $j = 1, \ldots, n$, be a finite collection of complex numbers. Then a *difference polynomial in $w(z)$* is a function which is polynomial in $w(z + c_j)$, $j = 1, \ldots, n$, with meromorphic coefficients $a_\lambda(z)$ such that $T(r, a_\lambda) = S(r, w)$ for all λ. The following theorem [48] is a difference analogue of the Clunie lemma [23].

Theorem 5.7 [48]. *Let $w(z)$ be a non-constant finite-order meromorphic solution of*

$$w(z)^n P(z, w) = Q(z, w),$$

where $P(z, w)$ and $Q(z, w)$ are difference polynomials in $w(z)$. If the degree of $Q(z, w)$ as a polynomial in $w(z)$ and its shifts is at most n, then

$$m(r, P(z, w)) = S(r, w).$$

In order to demonstrate exactly how theorem 5.7 can be used to obtain information about the density of poles of finite-order meromorphic solutions of difference equations, we consider as an example the following difference Painlevé equation

$$\overline{w} + \underline{w} = \frac{\alpha z + \beta}{w} + \frac{\gamma}{w^2} \tag{76}$$

with constant parameters α, β, γ. Recall that we have suppressed the z-dependence by writing $w \equiv w(z)$, $\overline{w} \equiv w(z+1)$ and $\underline{w} \equiv w(z-1)$. Suppose that w is a non-rational meromorphic function of finite order. Then by considering (76) in the form

$$w^2(\overline{w} + \underline{w}) = (\alpha z + \beta)w + \gamma$$

we may apply theorem 5.7 with $P(z, w) = \overline{w} + \underline{w}$ and $Q(z, w) = (\alpha z + \beta)w + \gamma$ thus obtaining

$$m(r, \overline{w} + \underline{w}) = S(r, w). \tag{77}$$

Since $T(r, \overline{w} + \underline{w}) = 2T(r, w) + O(\log r)$ by (52) and (76), equation (77) yields

$$N(r, \overline{w} + \underline{w}) = 2T(r, w) + S(r, w). \tag{78}$$

Finally, since $N(r, \overline{w} + \underline{w}) \leqslant 2N(r+1, w) = 2N(r, w) + S(r, w)$ (see (50) for the latter equality) we conclude by equation (78) that

$$N(r, w) = T(r, w) + S(r, w). \tag{79}$$

In particular, this implies that all non-rational finite-order meromorphic solutions of the difference Painlevé equation (76) have infinitely many poles.

Just as theorem 5.7 can be used to study the pole distribution of finite-order meromorphic solutions of difference equations, the following theorem enables the analysis of the value distribution of solutions for finite values. It is an analogue of a result due to A Z Mohon'ko and V D Mohon'ko [90] on differential equations.

Theorem 5.8 [48]. *Let $w(z)$ be a non-constant finite-order meromorphic solution of*

$$P(z, w) = 0, \tag{80}$$

where $P(z, w)$ is difference polynomial in $w(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function $a \in \mathcal{S}(w)$, then

$$m\left(r, \frac{1}{w - a}\right) = S(r, w).$$

Consider again the difference Painlevé equation (76) as an example. If the parameter α is non-zero, then (76) does not have any constant solutions. Therefore theorem 5.8 yields

$$m\left(r, \frac{1}{w - a}\right) = S(r, w)$$

for all $a \in \mathbb{C}$. This immediately implies that all non-rational finite-order meromorphic solutions of the non-autonomous equation (76) have infinitely many a-points for all $a \in \mathbb{C}$.

5.3. The proof of theorem 5.5

Within the proof of theorem 5.5 we often say that there are at most $S(r, w)$ points z_0 with a certain property. By this we mean that the integrated counting function $N(r, \cdot)$ measuring the points with the property in question is at most of the growth $S(r, w)$. In loose terms this means that there are very few points with this property. Conversely, we use expressions like: 'There are more than $S(r, w)$ points such that . . . '. In precise terms, this means that

$$\limsup_{r \to \infty} \frac{N(r, \cdot)}{T(r, w)} = c > 0,$$

where $c \in \mathbb{R}^+ \cup \{\infty\}$, and r runs to infinity in a set with infinite logarithmic measure. Intuitively speaking, this expression means that there exist a relatively large number of points with the property in question. For instance, by defining

$$M := \{\zeta : a_\pm, c_\pm, c_2 \text{ are analytic at } \zeta \text{ and } c_2(\zeta) \neq 0\}$$

we have that $\mathbb{C} \setminus M$ consists of at most $S(r, w)$ points.

Let w be a meromorphic solution of (72) and choose a point $z_0 \in M$ such that $w(z_0) = a_\pm(z_0)$ with multiplicity k. Then by (72) $w(z_0 + 1) = \infty$ with multiplicity l and $w(z_0 - 1) = \infty$ with multiplicity $k - l$ (and it may happen that either $k = 0$ or $k - l = 0$ which just means that either $w(z_0 + 1)$ or $w(z_0 - 1)$ is finite, respectively). We define the *zero-$a_\pm$ sequence of w at z_0* to be the longest possible list of points $L(z_0, w) = (z_{-m}, \ldots, z_0, \ldots, z_n)$, where $z_j = z_0 + j$, such that $w(z_{2j}) = a_\pm(z_{2j})$ at all $z_{2j} \in L(z_0, w)$ and $w(z_{2j+1}) = \infty$ at all $z_{2j+1} \in L(z_0, w)$. The following lemma is an immediate consequence of equation (72) and the definition of $L(z_0, w)$.

Lemma 5.9. *Let w be a meromorphic solution of equation (72) such that $w(z_0) = a_\pm(z_0)$, $z_0 \in M$, with multiplicity k. Then either*

(a) $L(z_0, w)$ *consists of at least three points, of which at least two are zeros of* $w - a_\pm$*; or*
(b) *the total number of zeros of* $w - a_\pm$ *(counting multiplicities) in* $L(z_0, w)$ *divided by the total number of poles of* w *is at most 1.*

When applying lemma 5.9 to study meromorphic solutions of (72) it is crucial to know that $w - a_\pm$ has sufficiently many zeros. Since (72) is irreducible, it is clear that none of the coefficients $a_\pm, c_\pm$ is a solution of the equation (72). Therefore, by theorem 5.8, we have $m\left(r, \frac{1}{w-a_\pm}\right) = S(r, w)$ and $m\left(r, \frac{1}{w-c_\pm}\right) = S(r, w)$ which implies that $N\left(r, \frac{1}{w-c_\pm}\right) = T(r, w) + S(r, w)$. Also, by applying theorem 5.7 with $P(z, w) = \overline{w}\underline{w}$ and $Q(z, w) = c_2 w^2 + c_1 w + c_0 - (aw + b)\overline{w}\underline{w}$, we obtain $m(r, \overline{w}\underline{w}) = S(r, w)$. Therefore, by (47), (48) and theorem 5.6, we have

$$m(r, w) \leqslant \frac{1}{2}m(r, \overline{w}\underline{w}) + \frac{1}{2}m\left(r, \frac{w}{\overline{w}}\right) + \frac{1}{2}m\left(r, \frac{w}{\underline{w}}\right) = S(r, w), \tag{81}$$

which implies that $N(r, w) = T(r, w) + S(r, w)$. In other words, w has a large number of poles, and $w - a_\pm$ and $w - c_\pm$ have a large number of zeros (for both choices of $\pm$).

Clearly all zeros of $w - a_\pm$ (which are in M) are a part of some sequences $L(z_0, w)$. Let $N_b(r, w)$ be the integrated counting function for those poles of w which are a part of a type (*b*) sequence defined in the statement of lemma 5.9. If there are more than $S(r, w)$ such poles, then by the definition of 'more than $S(r, w)$' there exists a constant $c > 0$ such that

$$N_b(r, w) \geqslant cT(r, w)$$

for all r in a set E with infinite logarithmic measure. Hence, by (52), (72), (81) and lemma 5.9, we have

$$\begin{aligned}
2T(r, w) = T(r, \overline{w}\underline{w}) + S(r, w) &= N(r, \overline{w}\underline{w}) - N_b(r, \overline{w}\underline{w}) + N_b(r, \overline{w}\underline{w}) + S(r, w) \\
&\leqslant 2(N(r+1, w) - N_b(r+1, w)) + N_b(r+1, w) + S(r, w) \\
&\leqslant (2 - c)T(r+1, w) + S(r, w)
\end{aligned} \tag{82}$$

for all r in a set with infinite logarithmic measure. Therefore w is of infinite order by inequality (50).

We conclude that if w is of finite order, then there are at least $T(r, w) + S(r, w)$ points z_0 such that $w(z_0 + 1) = \infty$ and one of the four following relations is valid:

$$w(z_0) = a_-(z_0) \qquad \text{and} \qquad w(z_0 + 2) = a_+(z_0 + 2) \tag{83}$$

$$w(z_0) = a_+(z_0) \qquad \text{and} \qquad w(z_0 + 2) = a_-(z_0 + 2) \tag{84}$$

$$w(z_0) = a_+(z_0) \qquad \text{and} \qquad w(z_0 + 2) = a_+(z_0 + 2) \tag{85}$$

$$w(z_0) = a_-(z_0) \qquad \text{and} \qquad w(z_0 + 2) = a_-(z_0 + 2). \tag{86}$$

Since $N\left(r, \frac{1}{w-a_\pm}\right) = T(r, w) + S(r, w)$ holds true for both a_- and a_+, exactly one of the following four cases is valid:

 (i) There are more than $S(r, w)$ points such that both (85) and (86) hold.
 (ii) There are more than $S(r, w)$ points such that both (83) and (84) hold, and at most $S(r, w)$ points such that either (85) or (86) holds.
(iii) There are more than $S(r, w)$ points such that (84) holds, and at most $S(r, w)$ points such that (83), (85) or (86) holds.
(iv) There are more than $S(r, w)$ points such that (83) holds, and at most $S(r, w)$ points such that (84), (85) or (86) holds.

The transformation $w \to 1/w$ takes the equation (72) into the form

$$\overline{w}\underline{w} = \frac{\frac{a_+ a_-}{c_2 c_+ c_-}(w - 1/a_+)(w - 1/a_-)}{(w - 1/c_+)(w - 1/c_-)}, \tag{87}$$

which still of the form (72) but with different coefficients. Therefore there are at least $T(r, w) + S(r, w)$ points such that one of the following options is valid:

$$w(z_0) = 1/c_-(z_0) \qquad \text{and} \qquad w(z_0 + 2) = 1/c_+(z_0 + 2) \tag{88}$$

$$w(z_0) = 1/c_+(z_0) \qquad \text{and} \qquad w(z_0 + 2) = 1/c_-(z_0 + 2) \tag{89}$$

$$w(z_0) = 1/c_+(z_0) \qquad \text{and} \qquad w(z_0 + 2) = 1/c_+(z_0 + 2) \tag{90}$$

$$w(z_0) = 1/c_-(z_0) \qquad \text{and} \qquad w(z_0 + 2) = 1/c_-(z_0 + 2). \tag{91}$$

Thus exactly one of the following statements is valid simultaneously to one of the (i)–(iv):

(i) There are more than $S(r, w)$ points such that both (90) and (91) hold.
(ii) There are more than $S(r, w)$ points such that both (88) and (89) hold, and at most $S(r, w)$ points such that either (90) or (91) holds.
(iii) There are more than $S(r, w)$ points such that (89) holds, and at most $S(r, w)$ points such that (88), (90) or (91) holds.
(iv) There are more than $S(r, w)$ points such that (88) holds, and at most $S(r, w)$ points such that (89), (90) or (91) holds.

In what follows we shall see that the case in which both (ii) and (ii$'$) are both valid leads to the full difference Painlevé III equation (74). Cases in which (i) and (i$'$), (i) and (ii$'$) or (ii) and (i$'$) hold true, equation (72) is either reduced into (75), or into a special case of the difference Painlevé III equation. In all cases in which any of (iii), (iii$'$), (iv) or (iv$'$) is valid the finite-order meromorphic solution w satisfies a Riccati difference equation (73).

5.3.1. The difference Painlevé III. We consider first the case in which (i) and (i$'$) are valid. Since by assumption there are more than $S(r, w)$ points such that (85) holds, it follows by equation (72) that there are more than $S(r, w)$ points z_0 such that $c_2(z_0) = a_+(z_0 - 1)a_+(z_0 + 1)$, and so in fact $c_2(z) \equiv a_+(z - 1)a_+(z + 1)$ since $a_\pm \in \mathcal{S}(w)$. Similarly, $c_2(z) \equiv a_-(z - 1)a_-(z + 1)$, and since a_+ and a_- are of finite order, either $a_+(z) \equiv a_-(z)$ or $a_+(z) \equiv -a_-(z)$. Since the former equation is in contradiction with the assumptions of theorem 5.5 we have $a_+(z) \equiv -a_-(z)$. Similarly $c_+(z) \equiv -c_-(z)$, and so equation (72) can be written in the form

$$\overline{w}\underline{w} = \frac{\frac{a_+^2}{\overline{a}_+ \underline{a}_+ c_+^2}\left(w^2 - 1/a_+^2\right)}{\left(w^2 - 1/c_+^2\right)}. \tag{92}$$

By combining the assumption (i$'$) and equation (92) it follows that

$$\frac{\overline{a}_+}{\overline{c}_+} \cdot \frac{\underline{a}_+}{\underline{c}_+} = \frac{a_+^2}{c_+^2}, \tag{93}$$

which can be solved to obtain $a_+/c_+ = \mu\lambda^z$, where μ is an arbitrary finite-order periodic function with period 1, and λ is any non-zero complex number. Since the transformation $w \to a_+ w$ takes equation (72) into the form

$$\overline{w}\underline{w} = \frac{w^2 - c_+^2/a_+^2}{w^2 - 1} \tag{94}$$

we conclude that if, in the case (i) and (i′), equation (72) has a finite-order meromorphic solution, then (72) is reduced to

$$\overline{w}\underline{w} = \frac{w^2 - \mu\lambda^{2z}}{w^2 - 1}$$

which is a special case of the difference Painlevé III equation.

Consider now the case (ii) and (ii′). Similarly as in the case (i), the assumptions of the case (ii) together with equation (72) imply

$$c_2 = \underline{a}_+\overline{a}_- = \underline{a}_-\overline{a}_+. \tag{95}$$

Equation (95) may be solved to obtain that a_+/a_- is an arbitrary periodic function with period 2. Similarly, the transformed equation (87) with the assumption (ii′) yields

$$\frac{c_2 c_+ c_-}{a_+ a_-} = \underline{c}_+\overline{c}_- = \underline{c}_-\overline{c}_+ \tag{96}$$

which can be combined with (95) to obtain

$$\left(\frac{c_+ c_-}{a_+ a_-}\right)^2 = \frac{\overline{c}_+\overline{c}_-}{\overline{a}_+\overline{a}_-} \frac{\underline{c}_+\underline{c}_-}{\underline{a}_+\underline{a}_-}. \tag{97}$$

On one hand, by solving equation (97) we have

$$\frac{c_+ c_-}{a_+ a_-} = \mu\lambda^{2z}, \tag{98}$$

where μ is an arbitrary finite-order periodic function with period 1, and λ is a complex constant. On the other hand, by assumption (ii′),

$$w(z_0 + 1)w(z_0 - 1) = \frac{c_2(z_0)c_+(z_0)c_-(z_0)}{a_+(z_0)a_-(z_0)}$$

$$w(z_0 + 1)^2 - (c_+(z_0 + 1) + c_-(z_0 + 1))w(z_0 + 1) + c_+(z_0 + 1)c_-(z_0 + 1) = 0$$

$$w(z_0 - 1)^2 - (c_+(z_0 - 1) + c_-(z_0 - 1))w(z_0 - 1) + c_+(z_0 - 1)c_-(z_0 - 1) = 0$$

at more than $S(r, w)$ points z_0. By combining these equations and using (97), we obtain

$$\Bigg((c_+(z_0 + 1) + c_-(z_0 + 1))\frac{c_2(z_0)c_+(z_0)c_-(z_0)}{a_+(z_0)a_-(z_0)}$$

$$- (c_+(z_0 - 1) + c_-(z_0 - 1))c_+(z_0 + 1)c_-(z_0 + 1)\Bigg)w(z_0 - 1) = 0. \tag{99}$$

Since by assumption $w(z_0 - 1) = c_\pm(z_0 - 1)$ and since there are more than $S(r, w)$ points z_0 such that (99) holds, the coefficient of w in (99) must vanish identically. In addition, by using the fact that $c_2 = \overline{a}_+\underline{a}_-$, we obtain the equation

$$\frac{\underline{c}_+ + \underline{c}_-}{\underline{a}_-} \cdot \frac{\overline{c}_+\overline{c}_-}{\overline{a}_+\overline{a}_-} = \frac{\overline{c}_+ + \overline{c}_-}{\overline{a}_-} \cdot \frac{c_+ c_-}{a_+ a_-} \tag{100}$$

which, by taking (98) into account, can be solved to have

$$\frac{c_+ + c_-}{a_-} = -\delta'\lambda^z, \tag{101}$$

where δ' is an arbitrary periodic function with period 2, and λ is as in (98). Therefore equation (72) takes the form

$$\overline{w}\underline{w} = \frac{\overline{a}_+\underline{a}_-(w^2 + a_-\delta'\lambda^z w + a_+ a_-\mu\lambda^{2z})}{(w - a_+)(w - a_-)}.$$

By denoting $\gamma := a_-/a_+$ and $\delta := \gamma \delta'$ (recall that a_-/a_+ is periodic with period 2) and using the transformation $w \rightarrow a_+ w$, we obtain

$$\overline{w}\underline{w} = \frac{\gamma w^2 + \delta \lambda^z w + \gamma \mu \lambda^{2z}}{(w-1)(w-\gamma)}$$

which is the difference Painlevé III equation (74).

Suppose now that (i) and (ii′) are valid. Then we have already shown that $c_2 = \overline{a}_+ \underline{a}_+ = \overline{a}_- \underline{a}_-$ and that $a_+ = -a_-$. A similar calculation to that of above shows that (97) and (98) hold also in this case. However, instead of (100), we have the equation

$$\frac{\underline{c}_+ + \underline{c}_-}{\underline{a}_-} \cdot \frac{\overline{c}_+ \overline{c}_-}{\overline{a}_+ \overline{a}_-} = -\frac{\overline{c}_+ + \overline{c}_-}{\overline{a}_-} \cdot \frac{c_+ c_-}{a_+ a_-}$$

which has the solution

$$\frac{c_+ + c_-}{a_-} = -\delta' \, e^{i\pi z/2} \lambda^z$$

rather than (101). Therefore in this case equation (72) has the form

$$\overline{w}\underline{w} = \frac{w^2 + \delta \, e^{i\pi z/2} \lambda^z w + \mu \lambda^{2z}}{w^2 - 1}.$$

Consider finally the case (ii) and (i′). In this case (98) is still valid, and in addition we have $c_+ = -c_-$. Hence equation (72) becomes

$$\overline{w}\underline{w} = \frac{\gamma w^2 + \mu \lambda^{2z}}{(w-1)(w-\gamma)}$$

which is a special case of (74).

5.3.2. The difference Riccati equation.

Since cases (iii), (iii′), (iv) and (iv′) are very similar to each other, we are satisfied with presenting explicit details only in the case (iii). Note that it does not matter which of the cases (i′)–(iv′) is valid simultaneously to (iii), we end up with w satisfying a Riccati difference equation nevertheless.

We have already shown that all except at most $S(r, w)$ zeros of $w - a_\pm$ are in a sequence $L(z_0, w)$ containing at least two $a_\pm$-points of w. From the statement (iii) it follows that those sequences must contain *exactly* one a_+-point and one a_--point (otherwise there would be more than $S(r, w)$ points where one of the relations (83), (85) or (86) would be valid).

Although all $a_\pm$-points of w (which belong to the set M) are part of some sequence $L(z_0, w)$ there might be poles of w which are not. However, if there are more than $S(r, w)$ poles which are not a part of any $L(z_0, w)$ then w would be of infinite order by a similar calculation as in (82).

We have so far shown that all except at most $S(r, w)$ poles of w are part of a sequence $L(z_0, w) = (l, a_+, \infty, a_-, m)$ where l and m are finite values. Suppose that the pole in the sequence $L(z_0, w)$ is of multiplicity k and that either $w - a_+$ or $w - a_-$ has a zero of order strictly less than k in $L(z_0, w)$ (this happens if either l or m is zero). In such a sequence the number of zeros of $w - a_\pm$ (counting multiplicities) divided by the number of poles of w is strictly less than 2, say α. If there are more than $S(r, w)$ poles in sequences of this type then, similarly as in (82), there is a $c > 0$ such that

$$2T(r, w) \leqslant (2 - (2-\alpha)c) \, T(r+1, w) + S(r, w). \tag{102}$$

Since $2 - (2-\alpha)c < 2$ inequality (102) combined with (50) yields a contradiction. Thus in all except at most $S(r, w)$ sequences $L(z_0, w)$ the multiplicities of all points are the same. Hence,

$$U := (w - a_+)(\overline{w} - \overline{a}_-) \tag{103}$$

satisfies $N(r, U) = S(r, w)$. Also, $m(r, U) = S(r, w)$ by (81) and (103). We conclude that $T(r, U) = S(r, w)$ and so equation (103) is the Riccati difference equation (73).

6. Linear difference equations

It is well known that the linear first-order difference equation

$$w(z + 1) = a(z)w(z) \qquad (104)$$

is explicitly solvable in terms of exponential and gamma functions if the coefficient $a(z)$ is a rational function, see e.g. [82]. Whittaker [128] has shown that in the more general case where the coefficient $a(z)$ is a finite-order meromorphic function, equation (104) admits a meromorphic solution w such that $\rho(a) \leqslant \rho(w) \leqslant \rho(a) + 1$. Recently Chiang and Feng [20] showed that if $a(z)$ an entire function of finite order, then all solutions of (104) satisfy $\rho(w) \geqslant \rho(a) + 1$.

The behaviour of meromorphic solutions of higher order linear difference equations remains relatively unknown. The first step towards understanding the growth of meromorphic solutions of linear difference equations of arbitrary order was given by Ishizaki and Yanagihara [67] who studied entire solutions of the equation

$$a_n(z)\Delta^n w(z) + \cdots + a_1(z)\Delta w(z) + a_0(z)w(z) = 0 \qquad (105)$$

with polynomial coefficients. They showed that if an entire solution of (105) has relatively slow growth, then the order of the solution is a rational number expressible in terms of the degrees of the polynomial coefficients. The exact formulation of their result is as follows. The *Newton polygon* for (105) is defined as the convex hull of $\cup_{j=0}^n \{(x, y) : x \geqslant j, y \leqslant \deg a_{n-j}(z) - (n - j)\}$.

Theorem 6.1 (Ishizaki and Yanagihara [67]). *Let $w(z)$ be a non-rational entire solution of (105) and of order $\chi < 1/2$. Then*

$$\log M(r, w) = Lr^\chi(1 + o(1)),$$

where χ is a slope of the Newton polygon for equation (105), and $L > 0$ is a constant. In particular, $\chi > 0$.

Wiman–Valiron theory (see [56] for a review) is an efficient tool when studying the growth of entire solutions of differential equations. In order to prove theorem 6.1, Ishizaki and Yanagihara developed a difference analogue of the Wiman–Valiron method for slow-growing entire solutions of linear difference equations. They also give examples of entire solutions of (105) satisfying the conditions of theorem 6.1.

Another approach to study meromorphic solutions of linear difference equations by Chiang and Feng relies on logarithmic difference estimates of finite-order meromorphic functions [20]. This approach enabled Chiang and Feng to find a uniform lower bound for the order of growth of meromorphic solutions of large classes of linear difference equations.

Theorem 6.2 (Chiang and Feng [20]). *Let $w(z)$ be a meromorphic solution of*

$$a_n(z)w(z + n) + \cdots + a_1(z)w(z + 1) + a_0(z)w(z) = 0 \qquad (106)$$

where the coefficients $a_n(z), \ldots, a_0(z)$ are polynomials. If there exists an integer $m \in \{0, \ldots, n\}$ such that

$$\deg(a_m) > \max_{\substack{0 \leqslant j \leqslant n \\ j \neq m}} \{\deg(a_j)\}, \qquad (107)$$

then $\rho(w) \geqslant 1$.

Chiang and Feng also considered the case where the coefficients of (106) are entire functions of finite order satisfying a growth condition analogous to (107). In particular, they showed that if the growth of one of the coefficients dominates the growth of others, then the order of growth of all non-zero solutions is at least one higher than the order of the coefficient with maximal growth [20].

One of the key tools used in the proof of theorem 6.2 is the following pointwise logarithmic difference estimate, which is analogous to an estimate on the logarithmic derivative due to Gundersen [44].

Theorem 6.3 (Chiang and Feng [20]). *Let $w(z)$ be a meromorphic function of finite order σ, and let $\varepsilon > 0$. Then*

$$\exp(-r^{\sigma-1+\varepsilon}) \leqslant \left| \frac{w(z+1)}{w(z)} \right| \leqslant \exp(r^{\sigma-1+\varepsilon})$$

for all $r = |z|$ outside of an exceptional set with finite logarithmic measure.

Ruijsenaars has studied the first-order linear difference equation

$$F(z + ia/2) = \Phi(z)F(z - ia/2), \tag{108}$$

where $\Phi(z)$ is a meromorphic function satisfying certain regularity conditions, and $a > 0$ is the step size of the equation. He found a new representation of solutions of (108) with no zeros and poles in a strip parallel to the real axis, and satisfying a certain growth condition in the same strip. Ruijsenaars named solutions of this type as 'minimal'. Special difference equations within the class (108) give rise to minimal solutions which can be understood as generalized gamma functions of hyperbolic, trigonometric and elliptic type, the standard gamma function being of rational type. These generalized gamma functions can be used to express the scattering and weight functions of certain integrable quantum systems [116].

The following theorem due to Chiang and Ruijsenaars [21] contains as special cases some of the Ruijsenaars' earlier results on minimal solutions [116, 117].

Theorem 6.4 (Chiang and Ruijsenaars [21]). *Assume that $\psi(z)$ is a function that is analytic in $S_c := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < c\}$ for some $c > 0$ and that satisfies*

$$\psi(z) = O(|z|^\nu), \qquad z \in S_c, \qquad |z| \to \infty,$$

for some $\nu > -1$, uniformly in closed substrips of S_c. Then the function

$$h(z) := \frac{\pi}{2ia^2} \int_{-\infty}^{\infty} \frac{\psi(z-x)}{\cosh(\pi x/a)} \, dx, \qquad z \in S_c,$$

admits analytic continuation to $S_{c+a/2}$. Introducing

$$H(z) := \int_0^z h(w) \, dw, \qquad z \in S_{c+a/2},$$

there exists $\gamma \in \mathbb{C}$ such that the function $\kappa(z) := H(z) + \gamma z$ satisfies the difference equation

$$\kappa(z + ia/2) - \kappa(z - ia/2) = \psi(z), \qquad z \in S_c. \tag{109}$$

Moreover, we have the bound

$$\kappa(z) - \gamma z = O(|z|^{\nu+1}), \qquad z \in S_{c+a/2}, \qquad |z| \to \infty,$$

uniformly in closed substrips of $S_{c+a/2}$.

Minimal solutions of (108) can be obtained by applying an exponential transformation to equation (109). A large class of coefficients $\Phi(z)$ such that $\psi(z) = \log \Phi(z)$ (for a fixed

branch of the logarithm) satisfies the conditions of theorem 6.4 can be constructed by using Hadamard products [21].

As we mentioned earlier, Whittaker has shown that if $\Phi(z)$ is of finite order, then there exists a meromorphic solution F of (108) such that $\rho(F) \leqslant \rho(\Phi) + 1$. Whittaker's proof of the existence is non-constructive in the sense that the meromorphic solution $F(z)$ such that $\rho(F) \leqslant \rho(\Phi) + 1$ is not expressed explicitly in terms of the coefficient $\Phi(z)$. Chiang and Ruijsenaars showed that if the coefficient of (108) is a meromorphic function of finite order ρ, and if equation (108) admits a special type of minimal solution satisfying an asymptotic growth condition depending on ρ, then the order the minimal solution is at most $\rho + 1$ [21]. Their approach is constructive in the sense that the obtained minimal solution has an explicit representation in terms of the coefficient of (108).

7. Value distribution of differences of meromorphic functions

In this section we will present a number of recent results on the value distribution of shifts of finite-order meromorphic functions, which may prove to be important tools in the study of analytic difference equations in the future.

7.1. Zeros of differences of meromorphic functions

Bergweiler and Langley have shown that differences of relatively slow growing meromorphic functions behave asymptotically like their derivatives in large parts of the complex plane [11]. The following lemma is an example of their results of this type.

Lemma 7.1 (Bergweiler and Langley [11]). *Let* $n \in \mathbb{N}$. *Let* f *be non-rational and meromorphic of order less than 1 in the plane. Then*

$$\Delta^n f(z) \sim f^{(n)}(z) \qquad \text{as} \quad z \to \infty \text{ in } \mathbb{C} \backslash E_n,$$

where $E_n \subset \mathbb{C}$ *is a countable union of discs*

$$E_n = \bigcup_{j=1}^{\infty} B(b_j, r_j) \qquad \text{such that} \quad \lim_{j \to \infty} |b_j| = \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{r_j}{|b_j|} < \infty.$$

A combination of asymptotic relations between differences and derivatives, such as lemma 7.1, with methods used to study derivatives of meromorphic functions is a useful tool in the study of differences of meromorphic functions with slow growth. The proof of the following theorem, for instance, is based on lemma 7.1 and the standard Wiman–Valiron theory.

Theorem 7.2 (Bergweiler and Langley [11]). *Let* $n \in \mathbb{N}$. *Let* f *be a non-polynomial entire function of order* $\rho < 1/2$, *and set*

$$G(z) = \frac{\Delta^n f(z)}{f(z)}.$$

If G *is non-rational then* G *has infinitely many zeros. In particular if* f *has order less than* $\min\left\{\frac{1}{n}, \frac{1}{2}\right\}$ *then* G *is non-rational and has infinitely many zeros.*

7.2. Second main theorem of Nevanlinna theory

We will now take a short digression from differences of meromorphic functions in order to complete the introduction to Nevanlinna theory started in sections 4 and 5.2.

The second main theorem of Nevanlinna theory says that if $q \geqslant 3$ and $a_1, a_2, \ldots, a_q$ are distinct complex numbers, then

$$(q-2)T(r, f) \leqslant \sum_{j=1}^{q} N\left(r, \frac{1}{f - a_j}\right) - N_1(r, f) + S(r, f), \tag{110}$$

where $N_1(r, f) := 2N(r, f) - N(r, f') + N\left(r, \frac{1}{f'}\right)$ is a positive quantity measuring the number of multiple a-points, and

$$S(r, f) = o(T(r, f)), \tag{111}$$

except possibly for a finite length of r-values. In particular, if $f(z)$ is of finite order, the error term $S(r, f)$ grows at most like $O(\log r)$ without an exceptional set. For infinite order functions, however, in general the exceptional set in (110) cannot be deleted [55].

In loose terms, Nevanlinna's second main theorem implies that the counting function $N(r, a)$ must usually be much larger than the proximity function $m(r, a)$ in the sum (44). In order to measure the deviation from regular value distribution more precisely, Nevanlinna introduced the *deficiency*

$$\delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, a)}{T(r, f)}$$

and the *ramification index*

$$\theta(a, f) = \liminf_{r \to \infty} \frac{N(r, a) - \overline{N}(r, a)}{T(r, f)},$$

where $\overline{N}(r, a)$ is the integrated counting function for the a-points of $f(z)$, ignoring multiplicities. The point $a \in \mathbb{C}$ is called a *deficient value* of $f(z)$ if $\delta(a, f) > 0$, and a *ramified value* of $f(z)$ if $\theta(a, f) > 0$. Moreover, it is said that a is *completely ramified* if all a-points of $f(z)$ have multiplicity 2 or higher.

Clearly, the deficiency satisfies $0 \leqslant \delta(a, f) \leqslant 1$ for all $a \in \mathbb{C} \cup \{\infty\}$. Moreover, by the second main theorem (110) it follows that for any meromorphic function $f(z)$ there can be at most countably many deficient values [54], and

$$\sum_{a \in \mathbb{C} \cup \{\infty\}} (\delta(a, f) + \theta(a, f)) \leqslant 2. \tag{112}$$

Inequality (112) is a deep generalization of Picard's theorem, which, translated to the language of Nevanlinna theory, says only that the counting function $N(r, a)$ may be identically zero for at most two values a in the extended complex plane $\mathbb{C} \cup \{\infty\}$. Borel's theorem states, on the other hand, that for any entire finite-order function $f(z)$ the deficiency $\delta(a, f)$ may be identically zero for at most two values a. Therefore the second main theorem is not only a generalization of Borel's result to meromorphic functions, but it tells much more on the value distribution of entire functions as well.

A natural problem connected to relation (112) is to find a meromorphic function $f(z)$ which at prescribed points has certain non-zero deficiencies $\delta(a, f)$ and ramification indices $\theta(a, f)$. This is the *inverse problem* for the deficiency relation, originally proposed and partially solved by Nevanlinna himself (see, for instance, [93]). The problem was finally completely solved by Drasin [26] by the means of quasi-conformal mappings.

7.3. Second main theorem and Picard's theorem for the shift operator

In this subsection we present a difference analogue of the second main theorem of Nevanlinna theory and some of its applications, including a Picard-type theorem for the shift operator. For proofs of the theorems and further discussion, see [47].

The following theorem is an analogue of the second main theorem (110), where the ramification term $N_1(r, f)$ has been replaced by a quantity depending on exact differences of a finite-order meromorphic function [47].

Theorem 7.3. *Let $c \in \mathbb{C}$, and let $f(z)$ be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Let $q \geqslant 2$, and let $a_1, \ldots, a_q$ be distinct complex constants. Then*

$$m(r, f) + \sum_{k=1}^q m\left(r, \frac{1}{f - a_k}\right) \leqslant 2T(r, f) - N_{\mathrm{pair}}(r, f) + S(r, f),$$

where

$$N_{\mathrm{pair}}(r, f) := 2N(r, f) - N(r, \Delta_c f) + N\left(r, \frac{1}{\Delta_c f}\right).$$

We will now discuss some of the implications of theorem 7.3. To this end, we assume that from now on $f(z)$ is a meromorphic function of finite order such that $f(z + c) \not\equiv f(z)$. We define $n_c(r, a), a \in \mathbb{C}$, to be the number of points z_0 in the disc of radius r centred at the origin such that $f(z_0) = a$ and $f(z_0 + c) = a$, where each point is counted according to the number of equal terms in the beginning of Taylor series expansions of $f(z)$ and $f(z + c)$ in a neighbourhood of z_0. We call such points *c-separated a-pairs* of $f(z)$ in the disc $\{z : |z| \leqslant r\}$. For instance, if in a neighbourhood of z_0,

$$f(z) = a + c_1(z - z_0) + \alpha(z - z_0)^2 + O((z - z_0)^3)$$

and

$$f(z + c) = a + c_1(z - z_0) + \beta(z - z_0)^2 + O((z - z_0)^3)$$

where $\alpha \neq \beta$, then the point z_0 is counted two times in $n_c(r, a)$. The number of *c-separated pole pairs*, $n_c(r, \infty)$, of $f(z)$ is the same as the number of c-separated 0-pairs of $1/f(z)$. This means that if $f(z)$ has a pole with multiplicity p at z_0 and another pole with multiplicity q at $z_0 + c$ then this pair is counted $\min\{p, q\} + m$ times in $n_c(r, \infty)$, where m is the number of equal terms in the beginning of the Laurent series expansions of $f(z)$ and $f(z + c)$ in a neighbourhood of z_0.

Note that since $f(z)$ was assumed not to be a periodic function with period c, it follows that $n_c(r, a)$ is finite for any finite r. Otherwise there would be a point $z_0 \in \mathbb{C}$ in a neighbourhood of which the series expansions of $f(z)$ and $f(z + c)$ would be identical. But this means that $f(z) \equiv f(z + c)$ in the whole complex plane, which contradicts the non-periodicity of $f(z)$. However, it is possible that $n_c(r, a)$ is strictly greater than the counting function $n(r, a)$.

We now define the integrated counting function for the c-separated a-pairs as

$$N_c(r, a) := \int_0^r \frac{n_c(t, a) - n_c(0, a)}{t} \, \mathrm{d}t + n_c(0, a) \log r, \tag{113}$$

where $a \in \mathbb{C} \cup \{\infty\}$. A natural difference analogue of $\overline{N}(r, a)$ is then

$$\widetilde{N}_c(r, a) := N(r, a) - N_c(r, a)$$

which, roughly speaking, counts the number of those a-points of $f(z)$ which are not c-separated pairs. It is immediately clear by definition that the inequalities $0 \leqslant \overline{N}(r, a) \leqslant T(r, f)$ and $\widetilde{N}_c(r, a) \leqslant T(r, f)$ are valid for any meromorphic function $f(z)$ and any $a \in \mathbb{C} \cup \{\infty\}$, as long as r is sufficiently large. However, the counting function $\widetilde{N}_c(r, a)$ may, for some values a, be negative for all r large enough. For instance, consider the function $g(z) := \wp(z) + \exp(z)$ where $\wp(z)$ is a Weierstrass elliptic function with a period $c \neq 2\pi \mathrm{i}$. Then $T(r, g) = N(r, g) + S(r, g)$ and each pole of $g(z)$ contributes 2 to $n(r, g)$ but -2 to

$\widetilde{n}_c(r, g)$. Therefore $\widetilde{N}_c(r, g) = -T(r, g) + S(r, g)$. The following theorem yields a lower bound for the growth of the counting function $\widetilde{N}_c(r, a)$ and implies that $\widetilde{N}_c(r, a)$ must be relatively large for most values $a \in \mathbb{C} \cup \{\infty\}$.

Theorem 7.4. *Let $c \in \mathbb{C}$, and let $f(z)$ be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Let $q \geqslant 2$, and let $a_1, \ldots, a_q$ be distinct complex constants. Then*

$$(q - 1)T(r, f) \leqslant \widetilde{N}_c(r, f) + \sum_{k=1}^{q} \widetilde{N}_c \left(r, \frac{1}{f - a_k} \right) + S(r, f).$$

Theorem 7.4 implies that $\widetilde{N}_c(r, a) \geqslant -T(r, f) + S(r, f)$ for all $a \in \mathbb{C} \cup \{\infty\}$, and $\widetilde{N}_c(r, a) = T(r, f) + S(r, f)$ for all except at most countably many values $a \in \mathbb{C} \cup \{\infty\}$ (see, for instance, [54] for a method of proving the latter relation). This means, by the definition of $\widetilde{N}_c(r, a)$, that points appearing pairwise, or in infinite lines, separated from each other by the constant vector $c \in \mathbb{C}$, are in a sense exceptional. For instance, by theorem 7.4 any finite-order meromorphic function $f(z)$ is either periodic with period c, or it can have at most one non-deficient value a such that whenever $f(z) = a$ also $f(z + c) = a$ and the first two terms in the series expansions of $f(z)$ at z and $z + c$ are identical (such as the function $g(z) = \wp(z) + \exp(z)$ discussed above).

In order to analyse this phenomenon more precisely, we introduce some additional notation. A difference analogue of the index of multiplicity $\theta(a, f)$ is called the *c-separated pair index*, and it is defined by

$$\pi_c(a, f) := \liminf_{r \to \infty} \frac{N_c(r, a)}{T(r, f)},$$

where $a \in \mathbb{C} \cup \{\infty\}$. Similarly, we define

$$\Pi_c(a, f) := 1 - \limsup_{r \to \infty} \frac{\widetilde{N}_c(r, a)}{T(r, f)},$$

which is an analogue of

$$\Theta(a, f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a)}{T(r, f)} \tag{114}$$

in the usual value distribution theory. With this notation in hand, theorem 7.4 yields the following analogue of the deficiency relation (112).

Corollary 7.5. *Let $c \in \mathbb{C}$, and let $f(z)$ be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Then $\Pi_c(a, f) = 0$ except for at most countably many $a \in \mathbb{C} \cup \{\infty\}$, and*

$$\sum_{a} (\delta(a, f) + \pi_c(a, f)) \leqslant \sum_{a} \Pi_c(a, f) \leqslant 2. \tag{115}$$

In the classical Nevanlinna theory it is clear by (114) that $0 \leqslant \Theta(a, f) \leqslant 1$ for all meromorphic functions $f(z)$ and for all a in the extended complex plane. For the difference analogue $\Pi_c(a, f)$, however, it is not immediately clear that a uniform upper bound even exists. The fact that $\Pi_c(a, f) \leqslant 2$ for all a follows by corollary 7.5, and the maximal deficiency sum

$$\sum_{a} \Pi_c(a, f) = 2 \tag{116}$$

may as a matter of fact be attained by a single value a. For instance, the function $g(z) = \wp(z) + \exp(z)$, where $\wp(z)$ is a Weierstrass elliptic function with a period $c \neq 2\pi i$, satisfies $\Pi_c(\infty, g) = 2$.

We say that a is an *exceptional paired value* of $f(z)$ *with the separation* c if the following property holds for all except at most finitely many a-points of $f(z)$: *whenever* $f(z) = a$ *then also* $f(z + c) = a$ *with the same or higher multiplicity*. Now, in a similar way as the second main theorem of Nevanlinna theory implies Picard's theorem, theorem 7.4 yields a shift analogue of Picard's theorem.

Corollary 7.6. *If a finite-order meromorphic function* $f(z)$ *has three exceptional paired values with the separation* c*, then* $f(z)$ *is a periodic function with period* c*.*

Since by definition all Picard exceptional values of $f(z)$ are also exceptional paired values, corollary 7.6 in fact yields the classical Picard's theorem for finite-order meromorphic functions. Corollary 7.6 also implies that if a finite-order meromorphic function $f(z)$ has two non-empty groups of three exceptional paired values with two different separations independent over the reals, say c_1 and c_2, then either $f(z)$ is a constant or $f(z)$ is an elliptic function with periods c_1 and c_2.

Concerning the sharpness of corollary 7.6, consider the elliptic function $sn(z, k)$, where $k \in (0, 1)$ is the elliptic modulus and K is the complete elliptic integral. The function $sn(z, k)$ is periodic with the periods $4K$ and $2iK'$, and it attains the value zero at points $2nK + 2miK'$ and has its poles at $2nK + (2m + 1)iK'$, where $n, m \in \mathbb{Z}$. Therefore $sn(z, k)$ has exactly two exceptional paired values with the separation $2K$, which is the maximal amount by corollary 7.6. Also, the function $f(z) = sn(z, k)$ is another example of a finite-order meromorphic function satisfying (112).

The restriction to finite-order meromorphic functions in corollary 7.6 (or corollary 7.5) cannot be removed, since $\exp(\exp(z))$ has three exceptional paired values with the separation $\log 2$. In addition to the Picard exceptional zeros and poles, the value 1 is exceptionally paired.

Analogously to complete ramification, we say that a point a is *completely paired with the separation* c if whenever $f(z) = a$ then either $f(z + c) = a$ or $f(z - c) = a$, with the same multiplicity.

Corollary 7.7. *Let* $c \in \mathbb{C}$*, and let* $f(z)$ *be a meromorphic function of finite order such that* $\Delta_c f \not\equiv 0$*. Then* $f(z)$ *has at most four completely paired points with separation* c*.*

As in corollary 7.7, a non-periodic finite-order function $f(z)$ can have at most three values a which only appear in lines of three (i.e. such that for some $z_0 \in \mathbb{C}$, $f(z_0) \neq a$, $f(z_0 + jc) = a$ with the same multiplicity for each $j = 1, 2, 3$, and $f(z_0 + 4c) \neq a$), and a maximum of two values which appear only in lines of four or more.

7.4. Meromorphic functions sharing values

Another consequence of Nevanlinna's second main theorem is the five value theorem, which says that if two non-constant meromorphic functions $f(z)$ and $g(z)$ share five values ignoring multiplicity then these functions must be identical. Nevanlinna also showed that if $f(z)$ and $g(z)$ are distinct, they can share four values *counting multiplicities* only in some very special cases which can be listed explicitly. For further studies concerning functions sharing three or four values see, for instance [42, 43, 91, 123].

By ignoring paired points instead of multiplicity, we obtain a difference analogue of the five value theorem. To be exact, we say that two meromorphic functions $f(z)$ and $g(z)$

share a point a, *ignoring c-separated pairs*, when $f(z) = a$ if and only if $g(z) = a$ with the same multiplicity, unless a is a c-separated pair of $f(z)$ or $g(z)$. In short, all paired points are ignored when determining whether or not $f(z)$ and $g(z)$ share a. This also means that if $f(z)$ has a paired a-point at z_0 and $g(z)$ has a single a-point at the same location, this point is not shared by $f(z)$ and $g(z)$.

Theorem 7.8. *Let $c \in \mathbb{C}$, and let $f(z)$ and $g(z)$ be meromorphic functions of finite order. If there are five distinct points a_k such that $f(z)$ and $g(z)$ share a_k, ignoring c-separated pairs, for all $k = 1, \ldots, 5$ then either $f(z) \equiv g(z)$ or both $f(z)$ and $g(z)$ are periodic with period c.*

The elliptic functions $\operatorname{sn} z$ and $1/\operatorname{sn} z$ share the points 1 and -1, counting multiplicities, and for both functions zero and infinity are exceptional paired values with separation $2K$. Therefore, $\operatorname{sn} z$ and $1/\operatorname{sn} z$ share the points $-1, 0, 1$ and ∞ ignoring pairs, which shows that the number five cannot be replaced by four in theorem 7.8.

8. q-difference equations

By a q-difference equation we mean an equation of the form $P(z, f) = 0$, where $P(z, f)$ is polynomial in f and finitely many of its q-shifts $f(q^n z)$, $n \in \mathbb{Z}$. The study of meromorphic solutions of q-difference equations has been ongoing since the 19th century. Valiron has shown that the non-autonomous Schröder q-difference equation

$$f(qz) = R(z, f(z)), \tag{117}$$

where $R(z, f(z))$ is rational in both arguments and nonlinear in $f(z)$, has a one parameter family of meromorphic solutions, if $q \in \mathbb{C}$ is suitably chosen [125]. Gundersen *et al* [45] showed that all meromorphic solutions of (117) are of order $\log_q(\deg_f R)$, where $\log_q$ denotes the q-based logarithm. Their result implies a q-difference analogue of a classical result due to Malmquist [85]. If the q-difference equation (117) admits a meromorphic solution of order zero, then (117) reduces to a q-difference Riccati equation, i.e. $\deg_f R = 1$.

There are a number of effective methods from different areas of complex analysis which can be applied to study the value distribution of meromorphic solutions of the Schröder equation (117), and q-difference equations in general. For instance, Eremenko and Sodin [28] used methods from complex dynamics to show that the Valiron and Nevanlinna deficient values of meromorphic solutions of the autonomous Schröder equation (117) always coincide with the exceptional values of $R(z)$. On the other hand, Ishizaki and Yanagihara [66] showed that this is not true in general for the non-autonomous Schröder equation. They have also studied Borel and Julia directions of meromorphic solutions of the Schröder equation by using methods from Nevanlinna theory [68, 69].

Just as equation (117) admits meromorphic solutions only for a special choice of q, the coefficients of a linear q-difference equation must satisfy certain conditions to guarantee the existence of meromorphic solutions. Bergweiler, Ishizaki and Yanagihara found sufficient conditions for the existence, and characterized the growth of meromorphic solutions of linear q-difference equations in terms of the Nevanlinna characteristic [9]. They concluded that all meromorphic solutions $f(z)$ of a linear q-difference equation with rational coefficients satisfy $T(r, f) = O((\log r)^2)$ and $(\log r)^2 = O(T(r, f))$. For entire solutions there are other methods, for instance, a q-shift analogue of the Wiman–Valiron theory by Bergweiler, Ishizaki and Yanagihara [10], and a functional analytic method by Ramis [112], by which the asymptotic behaviour of solutions may be characterized even more precisely.

The zero distribution of entire solutions of linear q-difference equations has been studied by Bergweiler and Hayman [8]. They showed that, under a certain condition, solutions behave

asymptotically like products of θ-functions, which enabled them to deduce an asymptotic formula for the locations of zeros. Heittokangas *et al* [59] studied the distribution of zeros and poles of meromorphic solutions of linear q-difference equations by using methods from Nevanlinna theory. Although their method does not give the locations of the poles and zeros explicitly, it can be used to obtain information on the relative densities of zeros and poles of solutions compared to the growth of the solution.

In section 5.2 we gave a number of results on Nevanlinna theory for finite-order meromorphic functions related to the shift operator $\Delta_c f = f(z+c) - f(z)$. Similar results may be obtained for the q-difference operator $\Delta_q f = f(qz) - f(z)$, provided that $f(z)$ is a meromorphic function of zero order [4]. The restriction to zero-order meromorphic functions is analogous to demanding finite order of growth in the ordinary shift case. For instance, all meromorphic solutions of linear and q-Riccati difference equation are of zero order. The following theorem is a q-shift analogue of theorem 5.6.

Theorem 8.1. *Let $f(z)$ be a non-constant zero-order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = o(T(r, f)) \tag{118}$$

for all r on a set E of logarithmic density 1, i.e. on a set E such that

$$\limsup_{r \to \infty} \frac{\int_{E \cap [0,r)} \frac{dt}{t}}{\log r} = 1.$$

Theorem 8.1 may be used to obtain a set of tools for analysing zero-order meromorphic solutions of q-difference equations in a similar fashion as theorem 5.6 in the case of difference equations. The procedure is similar to that already described in section 7 [4].

9. Future work and conclusions

Nevanlinna theory is only one of several approaches to the discrete/difference Painlevé equations. Being complex analytic in nature, it is very much in the spirit of the original Painlevé property for differential equations.

There are many important open problems in this area. We list some below.

 (i) Prove the existence of finite-order meromorphic solutions for difference Painlevé equations and characterize the remaining freedom in the periodic functions.
 (ii) Determine which classes of rational difference equations admit meromorphic solutions and characterize them (e.g., how many are there?)
(iii) Does the existence of sufficiently many finite-order meromorphic solutions guarantee the existence of a Lax pair?
(iv) Classify equations whose only finite-order meromorphic solutions are non-admissible.
 (v) Develop connections with other approaches (algebraic entropy, Diophantine integrability, affine Weyl algebras and Bäcklund transformations).
(vi) Study the existence of finite-order meromorphic solutions of higher dimensional (i.e. lattice) equations—e.g., dKdV, dKP, etc.
(vii) Determine in what sense the solutions of the difference Painlevé equations define new transcendents.

Acknowledgments

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